Real-Number Quantum Mechanics

A Structured Overview of the Literature, 1960–2026

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Abstract

Whether quantum theory *needs* complex numbers is one of the oldest questions in the subject and, as of 2026, one of the least settled. This document surveys the whole arc: Stueckelberg's 1960 real-Hilbert-space programme and the lattice-theoretic trichotomy that made $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ the only options; the pre-2021 consensus that real quantum theory could simulate everything; the 2021 Renou–Navascués no-go theorem and the network experiments that confirmed it within days; and the 2025–26 reversal, in which at least three independent groups constructed real-valued theories that reproduce *all* quantum predictions by relaxing the tensor-product composition rule. The central finding of this review is that the dispute is not empirical. Every party agrees on what the experiments measured. What is contested is which composition rule a real theory is entitled to adopt, and whether "independent sources" is a structural or an operational notion. The current live front is the attempt, by Renou and collaborators, to adjudicate this using fermionic information theory as an external constraint on candidate postulates.

Contents

1 The question, stated carefully

There is a trivial version of this question and a substantive one, and most confusion in the popular literature comes from conflating them.

The trivial version—"are complex numbers needed to describe experiments?"—has an immediate answer: no. Experimental outcomes are probabilities, hence real numbers. Renou *et al.* open their paper by conceding exactly this [1]. Complex numbers appear in classical electromagnetism too, where nobody regards them as ontological.

The substantive version is: *within a fixed theoretical framework, can the complex Hilbert spaces be replaced by real ones without loss of predictive power?* This only becomes sharp once the framework is pinned down. Renou *et al.* pin it down as four postulates [1]:

(1) To every physical system S there corresponds a Hilbert space $\mathcal{H}_S$, and its state is a normalised vector $\varphi \in \mathcal{H}_S$.

(2) A measurement is a family $\{\Pi_r\}_r$ of projectors on $\mathcal{H}_S$ with $\sum_r \Pi_r = \mathbb{I}_S$.

(3) Born rule: $\Pr(r) = \langle\varphi|\Pi_r|\varphi\rangle$.

(4) **Composition:** $\mathcal{H}_{ST} = \mathcal{H}_S \otimes \mathcal{H}_T$.

Writing ($1_\mathbb{C}$) for the version of postulate 1 with complex Hilbert spaces and ($1_\mathbb{R}$) for the real version, "complex quantum theory" (QT) is ($1_\mathbb{C}$) + (2)–(4), and "real quantum theory" (RQT) is ($1_\mathbb{R}$) + (2)–(4).

Everything in the modern debate turns on postulate (4). It is the only postulate that is not forced once you have chosen a number field, and it is the one that the 2025–26 counter-literature abandons. Renou *et al.* were explicit that their theorem does not touch formulations in which the composite Hilbert space is not a tensor product—they list path integrals, quasi-probability formulations, Wigner functions and Bohmian mechanics as lying outside scope—and they noted that "our results imply that this violation is in fact necessary for any such model" [1]. That sentence, read charitably, is the whole subsequent literature in embryo.

The founders' discomfort. The historical motivation is genuine and worth recording. Schrödinger to Lorentz, 6 June 1926:

> "What is unpleasant here, and indeed directly to be objected to, is the use of complex numbers. Ψ is surely fundamentally a real function."

Quoted in [1]; see also Schrödinger's 1926 papers and the Ehrenfest correspondence.

2 Mathematical background: why $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and nothing else

2.1 The lattice-theoretic trichotomy

The reason real quantum mechanics is a serious candidate rather than an idle variation is that the standard reconstruction routes do not single out $\mathbb{C}$. Starting from Birkhoff and von Neumann's quantum logic [8] and running through Piron's and Varadarajan's representation theorems [9, 10], one arrives at Solèr's theorem [7]: an orthomodular lattice satisfying the relevant regularity conditions is the lattice of closed subspaces of a Hilbert space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$—the three associative real division algebras—and over nothing else. Baez's survey [11] gives the cleanest account of why the division-algebra trichotomy is forced.

So the question "why $\mathbb{C}$?" is not rhetorical. Some *additional* physical input is required to eliminate $\mathbb{R}$ and $\mathbb{H}$, and identifying that input is what the whole field is about.

2.2 Stueckelberg's construction (1960)

Stueckelberg [3] showed that a d-dimensional complex quantum system can be embedded in a $2d$-dimensional real one, with the complex structure recovered by imposing a superselection rule: there exists a real antisymmetric operator J with $J^2 = -\mathbb{I}$ that commutes with all observables, and J plays the role of multiplication by i. Real quantum mechanics *plus* the superselection rule *is* complex quantum mechanics.

The construction Renou *et al.* use to make the same point at the level of states is compact enough to quote. For any complex density matrix ρ,

$$\tilde{\rho} \;=\; \mathrm{Re}(\rho) \otimes \frac{\mathbb{I}}{2} \;+\; \mathrm{Im}(\rho) \otimes \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;=\; \tfrac{1}{2}\big(\rho \otimes |{+i}\rangle\langle{+i}| \;+\; \rho^* \otimes |{-i}\rangle\langle{-i}|\big), \tag{1}$$

with $|{\pm i}\rangle = (|0\rangle \pm i|1\rangle)/\sqrt{2}$, is a legitimate real state, and with $\tilde{\Pi}_r = \Pi_r \otimes |i\rangle\langle i| + \Pi_r^* \otimes |{-i}\rangle\langle{-i}|$ one has $\mathrm{tr}(\rho\Pi_r) = \mathrm{tr}(\tilde{\rho}\tilde{\Pi}_r)$ exactly [1]. The auxiliary two-level system is the *rebit* that carries the phase.

2.3 The ubit model

Wootters reinterpreted Stueckelberg's superselection rule not as a constraint but as an object: a single, universal, non-localised real qubit—the *ubit* ("universal rebit")—to which every system in the world is coupled [4, 5]. Ordinary complex quantum mechanics is recovered if one restricts the allowed measurements and transformations on the ubit (Stueckelberg's rule), but Aleksandrova, Borish and Wootters attempted to recover the standard theory *without* imposing such a restriction, by studying the effective theory of a subsystem when the ubit interacts with a large environment [5]. The ubit programme is the most physically vivid version of "the imaginary unit is a dynamical degree of freedom."

2.4 The quaternionic cousin, and a symmetry argument for $\mathbb{C}$

Adler's monograph [12] is the standard reference for quaternionic quantum mechanics; it is worth reading alongside the real case because the obstructions are structurally similar (failure of local tomography, composition-rule pathologies) and better documented.

The most interesting *non*-Bell argument that $\mathbb{C}$ is forced comes from Moretti and Oppio [13, 14]. Working in Wigner's framework—elementary relativistic systems as irreducible, locally faithful, strongly continuous unitary representations of the Poincaré group—they prove that if the squared-mass operator is non-negative, the system admits a natural, Poincaré-invariant complex structure, unique up to sign, commuting with the whole observable algebra. Relativistic symmetry, not networks, does the work. The same argument runs in the quaternionic case [14].

This line deserves more attention than it gets. It is logically independent of the entire Renou debate, it does not rely on any composition postulate, and it applies exactly where the network arguments are weakest: relativistic field theory.

3 A taxonomy of "real quantum theories"

Much cross-talk in this literature is resolved by noticing that "real quantum theory" names at least four inequivalent things.

Variant	Composition rule	Local tomography?	Status
$\mathrm{RQT}_{\otimes}$	$\mathcal{H}_{ST} = \mathcal{H}_S \otimes_{\mathbb{R}} \mathcal{H}_T$	No	Falsified by [1] + experiments
Rebit/ubit encodings	$\otimes_{\mathbb{R}}$ plus one shared auxiliary rebit	Effectively yes	Reproduces all QT; needs a global reference [2, 5]
Non-tensorial real theories	Modified: $\mathcal{H}_{ST} \subsetneq$ or $\neq \mathcal{H}_S \otimes_{\mathbb{R}} \mathcal{H}_T$	Yes, by construction	Live; [30, 33, 36, 37, 38]
Real reformulations of QT	Wigner functions, path integrals, Bohmian mechanics	n/a	Never in scope of [1]

3.1 The structural deficit: local tomography

The single most important mathematical fact about $\mathrm{RQT}_{\otimes}$ is that it violates *local tomography*: the joint state of a bipartite system is not determined by the statistics of local measurements. Counting dimensions, a real d-dimensional system has a state space of dimension $d(d+1)/2$, while the real bipartite space $\mathbb{R}^d \otimes \mathbb{R}^d$ has $d^2(d^2+1)/2$ parameters—strictly more than the product of the local counts. Complex quantum theory is locally tomographic; real quantum theory has extra global degrees of freedom that no local measurement can see. Hardy and Wootters called this "limited holism" [6].

Local tomography is precisely the axiom that appears in the information-theoretic reconstructions of quantum theory—Hardy [15], Masanes and Müller [16], Chiribella, D'Ariano and Perinotti [17], Masanes *et al.* [18]—and it is what those derivations use to eliminate $\mathbb{R}$ and $\mathbb{H}$. This is worth stating plainly: **before 2021, the standard reason for preferring $\mathbb{C}$ was an axiom about tomography, not an experiment.** Renou *et al.*'s contribution was to convert a structural deficit into an operational one.

4 Before 2021: real quantum theory simulates everything

4.1 McKague, Mosca and Gisin (2009)

The reference result is [2]: any multipartite complex quantum experiment—states, discrete or continuous evolution, and measurements—can be simulated in a real Hilbert space using *local* real operations, at the cost of one additional rebit per party plus a shared reference. Earlier constructions had been limited to discrete evolution and could not keep local operations local. McKague *et al.* also worked out consequences for Bell inequalities and self-testing.

The upshot, and the reason the question was considered closed for a decade: in any *Bell* scenario, real quantum theory matches complex quantum theory. Even CHSH is no obstacle—the Tsirelson bound $2\sqrt{2}$ is attainable with real measurements on a real two-qubit state [1].

4.2 Why single-system experiments cannot decide it

A finite-dimensional single-site experiment *can* distinguish the theories, because $\mathrm{RQT}_{\otimes}$ is not locally tomographic. But Hilbert-space dimension cannot be bounded from above experimentally [19], and any real physical system has infinitely many degrees of freedom. Regina—Renou *et al.*'s personification of the real quantum physicist—is therefore entitled to reply to any such experiment by increasing the dimension. To defeat her you need an argument valid in *arbitrary*, including infinite, dimension [1]. This constraint is what forces the network construction.

5 The 2021 no-go theorem

5.1 The construction

Renou, Trillo, Weilenmann, Le, Tavakoli, Gisin, Acín and Navascués [1] use a standard entanglement-swapping network: two *independent* sources, three observers. Source 1 distributes a maximally entangled two-qubit state $\bar{\sigma}_{AB_1}$ to Alice and Bob; source 2 distributes $\bar{\sigma}_{B_2C}$ to Bob and Charlie. Bob performs a Bell-state measurement on his two particles, swapping entanglement onto Alice–Charlie. Alice and Charlie perform, respectively, three and six dichotomic measurements chosen to maximally violate

$$\mathrm{CHSH}_3 \;:=\; \mathrm{CHSH}(1,2;1,2) + \mathrm{CHSH}(1,3;3,4) + \mathrm{CHSH}(2,3;5,6) \;\leq\; 6, \tag{2}$$

a sum of three CHSH expressions [23, 24]. The maximal quantum value of (2) is $3\beta_{\mathrm{CHSH}} = 6\sqrt{2}$, attainable only with *complex* qubit measurements.

The figure of merit $\mathcal{T}$ is the average of the CHSH_3 violations over Bob's four outcomes, weighted by their probabilities. In the ideal realisation:

$$\mathcal{T}^{\mathrm{QT}}_{\max} = 6\sqrt{2} \approx 8.49, \qquad \mathcal{T}^{\mathrm{RQT}} \lesssim 7.66. \tag{3}$$

5.2 The proof technique and its status

The real bound in (3) is obtained by relaxing the problem to a convex optimisation over real quantum systems of arbitrary dimension, using the NPA hierarchy [20, 21, 22] and its network extensions, solved numerically. Two points deserve emphasis:

- **The bound 7.66 is an upper bound, and its tightness is explicitly left open by the authors** [1]. The gap $8.49 - 7.66$ is what the experiments must exploit.
- Because $\mathcal{T}$ is linear in the observed probabilities, the result survives shared randomness between the sources, which in turn permits dropping the i.i.d. assumption in multi-round hypothesis testing [1].

The required state visibility is $\sqrt{7.66/6\sqrt{2}} \approx 0.95$ [1]—demanding but routine—plus a two-qubit entangled measurement. The setup is essentially the bilocality scenario of Branciard, Gisin and Pironio [25], for which several implementations already existed.

5.3 What the theorem does and does not establish

Claim 1 (Established)**.** No assignment of real states and real measurement operators, on real Hilbert spaces of *any* dimension, composed by the real tensor product, and with the two sources modelled as a product state (possibly mixed with shared randomness), reproduces the observed network correlations.

Claim 2 (Not established)**.** That no real-valued formulation whatsoever can reproduce quantum predictions. Postulate (4) is a genuine assumption and the paper says so.

6 The experiments

Two experiments appeared essentially simultaneously with the Nature paper, in back-to-back *PRL* issues, followed by a strict-locality version.

Experiment	Platform	Reference	Result
Chen *et al.* (2022)	Superconducting processor, individual control and readout	PRL **128**, 040403 [26]	Entanglement swapping at fidelity 0.952; real bound 7.66 violated by $\sim 43\sigma$
Li *et al.* (2022)	Photonic network, SPDC polarisation entanglement, SNSPD detection	PRL **128**, 040402 [27]	Violation at $\sim 4.5\sigma$, in an optical quantum network
Wu *et al.* (2022)	Photonic, strict locality constraints	PRL **129**, 140401 [28]	Refutation under spacelike-separation conditions

The APS *Physics* coverage at the time was headlined "Quantum Mechanics Must Be Complex" [29]—a framing that, four years on, reads as premature.

Residual loopholes. Three matter, in descending order of importance:

1. **Source independence.** The theorem needs the two sources to be independent. Operationally this is enforced by physical separation and by using distinct pump lasers/crystals; theoretically it is imposed as a *product-state* assumption. The distinction between these two readings is the crux of the 2026 dispute (§8).
2. **Detection efficiency**, in the photonic implementations.
3. **Measurement independence / freedom of choice**, as in all Bell tests.

7 Early critiques, 2021–2024

Finkelstein (2021). The first response [30] made the point that would later be developed at length: the no-go applies only to real theories obeying the tensor-product rule. Finkelstein exhibits a real theory that does not obey it, and argues it permits no superluminal signalling because it is equivalent to complex quantum theory. The note is three pages and is the cleanest statement of the loophole.

The interpretive objection. A recurring line—see e.g. Gidney's widely read exposition [32]—is that the experiments test a conjunction (real amplitudes *and* real tensor composition), and a falsified conjunction does not identify which conjunct failed. Renou *et al.* would reply that postulate (4) is not negotiable because it holds for spacelike-separated systems in axiomatic QFT [1]. That reply is the real disagreement, and it is a disagreement about physics, not about logic.

Erba and Perinotti (2024): the composition rule is not unique. Logically prior to the whole 2025–26 wave, and not always cited by it, is [31]. Working operationally, Erba and Perinotti exhibit a family of theories differing from standard quantum theory precisely in the system-composition rule, while agreeing with it on all Bell-like correlation scenarios. This settles in the negative the long-standing question of whether the tensor-product postulate is the only consistent way to combine quantum systems. Whatever one concludes about real numbers specifically, this result removes the presumption that postulate (4) is forced—and that presumption was doing quiet work in the reception of [1].

Symmetry-based work. Moretti and Oppio [13, 14], discussed above, remain the strongest independent argument for $\mathbb{C}$ and are largely orthogonal to the network debate.

8 The 2025–2026 turn

Between March 2025 and June 2026 three independent groups published real-valued formulations reproducing all quantum predictions. All three relax postulate (4). They differ in how they motivate the replacement.

8.1 Barrios Hita, Trushechkin, Kampermann, Epping and Bruß

Quantum mechanics based on real numbers: a consistent description, arXiv:2503.17307; published as PRL **136**, 240202 (2026) [33], with an APS *Physics* viewpoint [34] and Quanta coverage [35].

Their move is methodological and, in my reading, the most defensible of the three: rather than *postulating* a composition rule, they start from a physical principle—*a local operation on one subsystem must not affect another, independently prepared subsystem*—and derive the composition rule that a real-valued representation must satisfy. Beginning from a specific real representation of single-system states and operations, and imposing that independent complex systems remain independent in the real encoding, they obtain a modified composition rule under which all multipartite predictions of standard quantum mechanics are recovered. Trushechkin's summary: they "simulate complex numbers by means of real numbers."

Conclusion drawn: real-valued quantum mechanics cannot be falsified, and the use of complex numbers is a matter of convenience.

8.2 Hoffreumon and Woods

Two papers, the second superseding the first in strength:

- *Quantum theory does not need complex numbers*, arXiv:2504.02808 (v2, October 2025) [36]. Constructs a real-number quantum theory consistent with the postulates, retaining *representation locality*—local physical operations are represented by local changes to the states. Concludes that real and complex quantum theory are experimentally indistinguishable.
- *Quantum theory based on real numbers cannot be experimentally falsified*, arXiv:2603.19208 (March 2026) [37]. The sharper result, and the one that names the crux directly.

The distinction Hoffreumon and Woods isolate is between

- **product-state independence**, which constrains the *mathematical form* of the source states, and
- **operational independence**, defined entirely by the *absence of observable cross-source correlations.*

Their theorem: once source independence is imposed *operationally*, every finite network correlation achievable in QT is also achievable in RQT with the same locality structure of measurements. They then extend the equivalence to arbitrary finite sequential multipartite protocols with channels and measurements of prescribed locality structure [37].

Their closing observation is the honest one: QT and RQT "support markedly different pictures of the correlation structure underlying the same observed world." The theories differ; the experiments cannot tell them apart.

8.3 Maioli, Curado and Gazeau

Quantum mechanics over real numbers fully reproduces standard quantum theory, arXiv:2604.19482 (April 2026, v3 June 2026) [38]. The most aggressive claim of the three: that the standard real tensor product used in the no-go theorems is *algebraically incompatible* with the structure of conventional quantum mechanics, so that the 2021 argument rests on an incomplete real formulation. They present a real-valued framework reproducing all predictions of standard quantum mechanics.

8.4 The reply: fermionic information theory

Kalarde, Xu and Renou, *Comment on "Quantum theory based on real numbers cannot be experimentally falsified": On the compatibility of physical principles with information theory for fermions*, arXiv:2604.07425 [39].

This is the current live front, and the argumentative move is elegant. The counter-literature all proceeds by proposing a *physically motivated postulate* that selects the right real composition rule. Kalarde *et al.* propose an external adequacy test for such postulates: *a general physical postulate ought to be satisfied by Fermionic Information Theory* (FIT), the standard framework for information encoded in the presence or absence of identical fermions. They then show the postulate of [37] fails in FIT, and hence is not general. They flag that the same confrontation deserves to be run against [33] and [36].

Note what has happened to the shape of the argument. It is no longer about experiments at all. It is about which *principles* a physical theory is entitled to assume, adjudicated by whether those principles survive contact with a framework—fermionic statistics—that everyone already accepts.

8.5 Adjacent 2025–26 work

- Elliott, *Strict advantage of complex quantum theory in a communication task*, PRA **111**, 062401 [40]: a communication task with provably lower cost in complex quantum theory than in classical or real quantum theory, with the necessity certified by geometric properties of state ensembles that witness basis-independent complexity. A genuinely different kind of argument from the network one.
- Giardino, *Deformed angular momentum algebra within the real Hilbert space*, arXiv:2603.07594 [41]: derives complex and quaternionic angular-momentum algebras from generalised position operators; the deformed algebras differ from the Hermitian ones but yield matching expectation values.
- Work on real-Hilbert-space dynamics and the harmonic oscillator [42, 43], and on entanglement, superselection and gauges in real quantum mechanics [44].

9 Imaginarity as a quantum resource

Running parallel to the foundational dispute, and largely unaffected by it, is a resource theory in which "imaginarity" is the resource, real states and real operations are free, and the usual machinery of monotones and conversion rates applies.

- Hickey and Gour [45] introduced the framework; Wu *et al.* established the operational theory [46] and the quantification and state-conversion theory [47], studying geometric imaginarity and robustness of imaginarity, with optical implementations counted in unfixed wave plates.
- Extensions: real quantum operations and state transformations [48], distributed scenarios [49], unitary-invariant witnesses [50], and the connection to Bargmann invariants [51].

The value of this line for the foundational question is that it makes "how much complexity does this state have?" a quantitative rather than a binary question, and it gives operational meaning to imaginarity independent of whether RQT is falsifiable. If the foundational debate ends in a stalemate—which currently looks likely—the resource theory is what survives as physics.

10 Where the debate actually stands

My reading of the state of play, stated as sharply as I can:

1. **The 2021 theorem is correct and the experiments are sound.** Nothing in the counter-literature disputes either. $\mathrm{RQT}_{\otimes}$ is dead.

2. $\mathrm{RQT}_{\otimes}$ **was never the interesting theory.** It fails local tomography, which was already the standard reason to reject it, and it was known since [2] that rebit encodings evade the problem at the cost of a shared reference. The 2021 result made this failure operational—a real achievement—but it did not broaden the target.

3. **The dispute is now about postulate (4), and it is not empirical.** Renou *et al.* defend the tensor product by appeal to axiomatic QFT for spacelike-separated systems. Their critics reply that a real theory should be permitted to derive its composition rule from a locality principle rather than inherit the complex one. Both are defensible positions about theory construction. No experiment discriminates. Erba and Perinotti's demonstration that consistent alternative composition rules exist [31] weakens the presumption in favour of the tensor product considerably.

4. **"Independent sources" is the pivot.** Hoffreumon and Woods' product-state versus operational independence distinction [37] identifies exactly where the theorem's assumption exceeds what an experiment can certify. If you can only verify the absence of cross-source correlations, you cannot verify the product-state form; and with the weaker assumption the separation disappears.

5. **The new adjudication criterion is principle-level, not experimental.** Kalarde, Xu and Renou [39] propose fermionic information theory as a proving ground for candidate postulates. This is the right move given that experiments have run out of discriminating power, and it is where the next few years of the argument will be fought.

6. **The strongest pro-$\mathbb{C}$ arguments are the least discussed.** Moretti and Oppio's Poincaré-symmetry result [13] and Elliott's communication-task separation [40] are both independent of the composition-rule dispute. If I were placing weight, I would put more on these than on the network programme.

A caution on framing. Both the 2022 headline "Quantum Mechanics Must Be Complex" [29] and the 2026 headline "Physicists take the imaginary numbers out of quantum mechanics" [35] overstate their respective results. What is actually true is narrower and more interesting: complex quantum theory and its real reformulations are empirically equivalent but structurally different, and choosing between them requires criteria that are not measurements.

11 Open problems

1. **Is the bound $\mathcal{T}^{\mathrm{RQT}} \lesssim 7.66$ tight?** Left open in [1] and still open as far as I can determine.

2. **Does any proposed real composition rule survive fermionic information theory?** [39] answers no for [37]; [33] and [36] have not been tested.

3. **Are the three 2025–26 real formulations the same theory?** They arrive by different routes at what may be one construction. Nobody appears to have compared them directly. This looks like the most tractable and most useful open question in the area.
4. **Do the real formulations extend to quantum field theory?** All are finite-dimensional and finite-network. Moretti and Oppio's result suggests they may not.
5. **Is there a network-free operational separation?** Elliott's communication task [40] is the closest thing; whether it too rests on a composition assumption deserves checking.
6. **What replaces local tomography?** If real theories with derived composition rules are admissible, the reconstruction programmes [16, 17] need a different axiom to single out quantum theory.

12 Annotated reading order

If the goal is to reach the research frontier efficiently, this is the order I would use. Total: roughly 25–30 hours to be able to argue with anyone in the field.

1. **Renou *et al.*, Nature 600, 625 (2021)** [1] — 3 h, including the Supplementary Information. Non-negotiable. Read the "Regina and Conan" framing carefully; it encodes the assumptions. Pay particular attention to the paragraph listing formulations out of scope—it is the seed of everything after.
2. **McKague, Mosca, Gisin, PRL 102, 020505 (2009)** [2] — 1.5 h. Read second, not first: it tells you what was already known, which is what makes the 2021 result surprising.
3. **Finkelstein, arXiv:2103.12740** [30] — 20 min. Three pages that state the loophole before anyone developed it.
4. **Hoffreumon and Woods, arXiv:2603.19208** [37] — 3 h. The sharpest of the counter-arguments, and the source of the product-state/operational distinction. If you read only one paper from the counter-literature, read this.
5. **Barrios Hita *et al.*, PRL 136, 240202 (2026)** [33] — 2 h, plus the *Physics* viewpoint [34] first as a 15-minute orientation.
6. **Kalarde, Xu, Renou, arXiv:2604.07425** [39] — 1 h. Short, and it defines the current front line.
7. **Moretti and Oppio, arXiv:1611.09029** [13] — 4 h, and mathematically the hardest item here. Worth it: it is the argument the network debate keeps not engaging with.
8. **Chen *et al.* and Li *et al.*, PRL 128** [26, 27] — 1.5 h for both. Skim for the systematics and the visibility budget rather than the conclusions.
9. **Wu *et al.*, PRL 126, 090401 (2021)** [46] — 2 h. Entry point to the imaginarity resource theory.
10. **Maioli, Curado, Gazeau, arXiv:2604.19482** [38] — 2 h. Read last among the counter-papers; the algebraic-incompatibility claim is the strongest and therefore the one most worth checking independently.

Background, as needed rather than in order. Baez [11] for the division-algebra trichotomy (excellent and readable); Varadarajan [10] or Piron [9] for the lattice-theoretic route and Solèr's theorem [7]; Adler [12] for the quaternionic analogue; Aleksandrova, Borish and Wootters [5] for the ubit model, which is the most physically suggestive thing in the older literature.

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