Reach for the $\star$: Quantum Mechanics Without Wavefunctions

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Abstract

Intended as a pedagogical resource for undergraduate students, this report provides a rigorous yet accessible introduction to the phase-space formulation of quantum mechanics. We explore the mathematical framework that reconciles the abstract operator algebra of Hilbert space with the intuitive geometric variables of classical mechanics. The Weyl-Wigner correspondence is established as the central dictionary, constructing a mapping where operators become Weyl symbols and quantum states are represented by Wigner quasi-probability functions. We derive the Moyal $\star$-product, a noncommutative deformation of pointwise multiplication that encodes the algebraic structure of quantum observables, and demonstrate how the Moyal bracket generates dynamics that reduce to the classical Poisson bracket in the semiclassical limit ($\hbar \to 0$). Finally, the quantum-classical transition is analyzed as a physical consequence of information loss; we argue that macroscopic coarse-graining acts as a low-pass filter on phase space, suppressing $\star$-product deformations and leading to the emergence of effective classicality.

Contents

1 Introduction

"Negative energies and probabilities should not be considered as nonsense. They are well-defined concepts mathematically, like a negative sum of money"

– P.A.M Dirac (Bakerian Lecture, 1942)

In 1942, while the world was engulfed in war, Paul Dirac stood before the Royal Society in London to deliver the Bakerian Lecture. By then, Dirac was a Nobel laureate who had successfully married quantum mechanics with special relativity, and quantum mechanics was no longer in its infancy. The Schrödinger equation had been written, the uncertainty principle established, and the rules of the quantum game seemingly set. Yet, Dirac, whose own equation had predicted antimatter, stood before his peers to defend a concept that seemed logically impossible: *negative* probability [1].

The need for negative probabilities arises from the mismatch between the two standard formulations of physics. Quantum mechanics is typically constructed in Hilbert space using wavefunctions $\psi(x)$ and non-commuting operators $\hat{A}$. In contrast, classical mechanics relies on the geometry of phase space, where a state is defined by a point (q, p) and observables are functions $A(q, p)$ governed by the Poisson bracket $\{A, H\}$.

This mismatch motivates a natural question:

Can quantum mechanics be expressed in a way that looks structurally closer to classical mechanics?

The phase-space approach does not claim that a particle literally has simultaneous sharp values of (q, p). Instead, it provides a different yet illuminating representation of the same physics, where quantum effects appear as a deformation of the algebra of classical observables.

This idea becomes valuable in topics where one wants to compare quantum and classical behavior, visualize quantum states, or model finite measurement resolution. The phase-space viewpoint has become standard in quantum optics and quantum state reconstruction, where Wigner- and Husimi-type distributions are heavily used to describe experiments.

1.1 Historical Route to Phase-Space Quantum Mechanics

The Phase Space formulation of quantum mechanics emerged through a decades-long struggle to reconcile the abstract operators of Hilbert space with the intuitive variables of classical physics.

In 1927, Hermann Weyl was interested in the correspondence principle, particularly the map between classical quantities (like x, p) and their quantum operator counterparts $(\hat{x}, \hat{p})$. Since quantum operators do not commute, this creates an ordering ambiguity when we try to turn a classical function into a quantum operator. The **Weyl Quantization** of a classical function $A(x, p)$ into an operator $\hat{A}$ is defined by the following integral transform:

$$\hat{A} = \frac{1}{(2\pi)^2} \iint dk\, dq\, \tilde{A}(q, k) e^{i(q\hat{x}+k\hat{p})}.$$

where $\tilde{A}(q, k)$ is the classical Fourier transform of the function $A(x, p)$ [9]. This ensures that real classical variables map to Hermitian quantum operators. However, at this stage, it was lacking a physical interpretation.

In 1932, Eugene Wigner was investigating quantum corrections to classical statistical mechanics (specifically, thermodynamic equilibrium). To calculate expectation values, he introduced the **Wigner quasi-probability distribution** $W(x, p)$. This object is not introduced as "probability of (q, p)" (which would contradict uncertainty), but as a mathematically precise way to represent quantum states on phase space. For a quantum state described by a wavefunction $\psi(x)$, the Wigner Function $W(x, p)$ is described as

$$W(x, p) = \frac{1}{\pi\hbar} \int_{\infty}^{\infty} \psi^*(x+y)\psi(x-y)e^{2ipy/\hbar}\, dy.$$

Unlike the wavefunction, $W(x,p)$ is always a real number. It acts like a probability distribution, but it can be negative in small regions of phase space. Therefore, for over a decade, the Wigner function was viewed merely as a computational trick, instead of a fundamental description of reality.

The connection to a standalone mechanics was forged in the 1940s through the work of **H.J. Groenewold** and **José Enrique Moyal**.

- In 1946, Groenewold [4] showed that one cannot build a quantization map that preserves all classical algebraic relations exactly (closely related to what later became known as the Groenewold-Van Hove obstruction). In particular, one cannot have a map that simultaneously sends Poisson brackets to commutators and preservrs products for all observables. This hinted that quantum mechanics must differ from classical mechanics at the algebraic level.
- In 1949, Moyal [7] proposed a formulation of quantum mechanics as a type of statistical dynamics on phase space. The cost of working with functions instead of operators is a modified product rule - the Moyal $\star$-product and a modified bracket, the Moyal bracket.

In compact form,

$$A \star B = A \exp\left[\frac{i\hbar}{2}\left(\overleftarrow{\partial_q}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_q}\right)\right] B..$$

which will be elaborated in the upcoming sections.

Moyal's framework was later placed into a broader mathematical program known as the **deformation quantization**, where quantum mechanics is viewed as the $\hbar$-deformation of the classical algebra of observables on a symplectic manifold, showed in the works of Bayen et al. [2].

Finally, it helps to know that the Wigner function is not the only phase-space representation. Different operator orderings lead to different quasi-probability distributions; the classic unifying treatment is presented in Cahill and Glauber [3].

1.2 Aim and Scope

The aim of this project is to build the phase-space formulation of quantum mechanics in a way that is rigorous enough but accessible to undergraduates. In particular, we want to investigate

1. State Representation: How does a quantum state ρ becomes a phase-space (Wigner) function $W_\rho(q,p)$
2. Algebra and Dynamics: How operator multiplication becomes the Moyal $\star$-product and how qunatum commutators become the Moyal bracket, with the classical limit visible in an $\hbar$-expansion.
3. The Operational Classical Limit: How the classical world emerges from the quantum one. We argue that the transition is not merely a mathematical limit $\hbar \to 0$, but a physical consequence of coarse-graining.

The rest of the report is structured as follows:

- Section 2 derives the Weyl symbol for observables and the Wigner function for states, culminating in the trace-pairing formula that allows quantum expectation values to be computed via classical-style integrals.
- Section 3 derives the deformed product law that replaces pointwise multiplication, show how it generates the equations of motion, and how it mimics classical dynamics.
- Section 4 models limited measurement resolution as a coarse-graining map. We demonstrate that this map acts as a low-pass filter in phase space, washing out the fine structures required for the $\star$-product.

2 The Weyl-Wigner Transform

"I simply can't resist playing the messenger—often an unwelcome one—in this drama between mathematics and physics."

– Hermann Weyl, Preface to *Gruppentheorie und Quantenmechanik* (1928)

By the end of Section 1 we have established that quantum mechanics *can* be written on phase space, but at the cost of a **noncommutative product** and a **quasi-probability** that may become negative. The goal of Section 2 is to make that promise precise. We will derive the Wigner function from a set of requirements that are hard to argue with once one decides on a phase-space formulation.

In classical statistical mechanics, a system (state) is described by a probability density $f(q,p)$ on phase space. Observables are functions $A(q,p)$, and expectation values are computed by

$$\langle A \rangle = \iint_{\mathbb{R}^2} A(q,p) f(q,p)\, dqdp, \quad \iint f(q,p)\, dqdp = 1.$$

In this case, the state is a weight function, the observable is a test function, and physics comes from integrating their product. Recall in quantum mechanics the expectation value is given by

$$\langle \hat{A} \rangle_{\text{qm}} = \mathrm{Tr}\left(\rho \hat{A}\right).$$

where ρ is the density operator and $\hat{A}$ is an observable operator. In the position basis,

$$\mathrm{Tr}\left(\rho \hat{A}\right) = \int_{\infty}^{\infty} \langle q | \rho \hat{A} | q \rangle\, dq.$$

This motivates the need of rewriting $\mathrm{Tr}\left(\rho \hat{A}\right)$ so that it looks like a classical phase-space average, which is an integral of a function for the observable multiplied by a function for the state. In short, we aim to construct

- a map $\hat{A} \mapsto A_W(q,p)$ (an operator $\to$ a phase-space symbol), and
- a map $\rho \mapsto W_\rho(q,p)$ (a state $\to$ a phase-space distribution)

such that

$$\mathrm{Tr}\left(\rho \hat{A}\right) = \iint_{\mathbb{R}^2} A_W(q,p) W_\rho(q,p)\, dqdp.$$

(up to a fixed normalization convention).

We will demand the following properties for the distribution (Wigner Function) W_ρ:

1. (Reality): For a physical (Hermitian) state ρ, the phase-space function $W_\rho(q,p)$ should be real.
2. (Normalization):
$$\iint W_\rho(q,p)\, dqdp = 1.$$
3. (Correct Marginals): Integrating out momentum gives the position probability density, and integrating out position gives the momentum probability density:
$$\int W_\rho(q,p)\, dp = \langle q | \rho | q \rangle, \quad \int W_\rho(q,p)\, dq = \langle p | \rho | p \rangle.$$
4. (Expectation-value pairing): There exists a ntural symbol map $\hat{A} \mapsto A_W$ such that
$$\mathrm{Tr}\left(\rho \hat{A}\right) = \iint A_W(q,p) W_\rho(q,p)\, dqdp.$$

Properties (1) and (2) impose standard statistical constraints. Property (3) ensures physical consistency: integrating out momentum must recover the correct position probability, $|\psi(q)|^2$. Property (4) establishes the function's utility, allowing quantum expectation values to be calculated via classical-style phase-space integrals. Crucially, we do not demand positivity ($W \geqslant 0$). A strictly positive distribution would imply the particle exists at a precise coordinate (q, p), with $W(q, p)$ reflecting our ignorance of its coordinates, which contradicts the Heisenberg uncertainty principle. Thus, we must accept $W(q, p)$ as a quasi-probability distribution.

These demands are not arbitrary: they formalize the minimum sense in which W_ρ behaves like a phase-space distribution while still reproducing quantum predictions. Reviews such as Hillery et al. [5] emphasize exactly this perspective and use it to motivate the standard formulas and their key properties.

At this stage it is tempting to assume that once we have $\hat{A} \mapsto A_W$, we can treat qunatum mechanics as classical mechanics on phase space, except with a funny-looking distribution W_ρ. Unfortunately, this is not true.

The trace pairing (requirement 4) can be made to work beautifully, at the cost of giving up ordinary multiplication. What replaces pointwise multiplication is exactly the object we will derive in Section 3, a new product $A_W \star B_W$ such that

$$(\hat{A}\hat{B})_W = A_W \star B_W.$$

The rest of this section will be dedicated in discussing the Weyl symbol (operators in phase-space) and the Wigner function. Most of the derivations are borrowed from [6].

2.1 The Weyl Symbol

In classical mechanics, an observable is a function $A(q, p)$ on phase space, and multiplication is just pointwise:

$$(AB)(q, p) = A(q, p)B(q, p).$$

In quantum mechanics, an observable is an operator $\hat{A}$, built from $\hat{q}, \hat{p}$ satisfying $[\hat{q}, \hat{p}] = i\hbar$. In this case, there exists an **ordering ambiguity** since the same classical product qp has at least two obvious operator candidates: $\hat{q}\hat{p}$ or $\hat{p}\hat{q}$, and neither choice is canonical.

A widely used resolution is Weyl ordering, which replaces each classical monomial by the fully symmetrized operator. For one degree of freedom, Weyl ordering sends

$$qp \mapsto \frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}).$$

which is the most symmetric choice. To generalize this idea, we wish to construct a map between operators $\hat{A}$ and phase-space functions $A_W(q, p)$ (called symbols).

We notice that the plane wave $\exp\left[\frac{i}{\hbar}(\xi\hat{x} + \eta\hat{p})\right]$ is the perfect building block since $\hat{x}$ and $\hat{p}$ are stuck together in the exponent, hence automatically symmetric.

Starting with an arbitrary classical phase-space function $A(q, p)$. By Fourier transform,

$$A(q, p) = \int\int d\xi d\eta \; \tilde{A}(\xi, \eta) e^{\frac{i}{\hbar}(\xi q + \eta p)}.$$

Using the symmetric physics convention (where $\hbar$ is in the exponent),

$$\tilde{A}(\xi, \eta) = \frac{1}{(2\pi\hbar)^2} \int\int dq' dp' \; A(q', p') e^{-\frac{i}{\hbar}(\xi q' + \eta p')}.$$

where $\frac{1}{(2\pi\hbar)^2}$ is the normalization coefficient (see Appendix A).

Now we apply the quantization rule. We assume the map is linear, so we simply replace the classical plane wave with the operator plane wave[1].

$$\exp\left[\frac{i}{\hbar}(\xi x+\eta p)\right] \mapsto \exp\left[\frac{i}{\hbar}(\xi \hat{x}+\eta \hat{p})\right].$$

So, the operator $\hat{A}$ is:

$$\hat{A}=\iint d\xi d\eta\ \tilde{A}(\xi,\eta)e^{\frac{i}{\hbar}(\xi\hat{q}+\eta\hat{p})}.$$

Now we substitute the definition of $\tilde{A}$:

$$\begin{aligned}\hat{A}&=\iint d\xi\, d\eta\left[\frac{1}{(2\pi\hbar)^2}\iint dq'dp'\ A(q',p')e^{-\frac{i}{\hbar}(\xi q'+\eta p')}\right]e^{\frac{i}{\hbar}(\xi\hat{q}+\eta\hat{p})}\\ &=\frac{1}{(2\pi\hbar)^2}\iiiint dq'dp'd\xi d\eta\ A(q',p')e^{-\frac{i}{\hbar}(\xi q'+\eta p')}e^{\frac{i}{\hbar}(\xi\hat{q}+\eta\hat{p})}.\end{aligned}$$

Since q',p' are just scalar numbers, they commute with the operators $\hat{q},\hat{p}$, so we can simply add the exponents:

$$-\frac{i}{\hbar}(\xi q'+\eta p')+\frac{i}{\hbar}(\xi\hat{q}+\eta\hat{p})=\frac{i}{\hbar}[\xi(\hat{q}-q')+\eta(\hat{p}-p')].$$

This gives us the final Weyl Quantization formula, in d dimensions,

$$\boxed{\hat{A}\equiv \mathrm{Op}_W[A]=\frac{1}{(2\pi\hbar)^{2d}}\int dqdpd\xi d\eta\ A(q,p)\exp\left\{\frac{i}{\hbar}\left[\xi(\hat{q}-q)+\eta(\hat{p}-p)\right]\right\}} \tag{2.1}$$

The Weyl Quantization eats in a symbol $A(q,p)$ on phase space and outputs an observable $\hat{A}$ on Hilbert space. In the position representation:

$$\langle x|\,\hat{A}\,|y\rangle=\frac{1}{(2\pi\hbar)^d}\int dp\ A\left(\frac{x+y}{2},p\right)e^{\frac{i}{\hbar}(x-y)p}.$$

for the sake of completeness, the derivation is included in the Appendix B.
For the inverse problem, we perform the variable change

$$q'=\frac{x+y}{2},\qquad s=x-y,$$

so that $x=q'+\frac{s}{2}$ and $y=q'-\frac{s}{2}$. Then the kernel formula becomes

$$K_{\hat{A}}\left(q'+\frac{s}{2},q'-\frac{s}{2}\right)=\left\langle q'+\frac{s}{2}\right|\hat{A}\left|q'-\frac{s}{2}\right\rangle=\frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}d^dp\ A(q',p)\,e^{\frac{i}{\hbar}s\cdot p}. \tag{2.2}$$

This is a Fourier transform of $A(q',p)$ with respect to p. We invert it by multiplying both sides by $e^{-\frac{i}{\hbar}p'\cdot s}$ and integrating over s:

$$\begin{aligned}\int_{\mathbb{R}^d}d^ds\ e^{-\frac{i}{\hbar}p'\cdot s}\left\langle q'+\frac{s}{2}\right|\hat{A}\left|q'-\frac{s}{2}\right\rangle&=\frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}d^dp\ A(q',p)\int_{\mathbb{R}^d}d^ds\ e^{\frac{i}{\hbar}s\cdot(p-p')}\\ &=\frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}d^dp\ A(q',p)\,(2\pi\hbar)^d\delta(p-p')\\ &=A(q',p').\end{aligned}$$

[1]By linearity, specifying the quantization map on the Fourier basis elements $\exp\left[\frac{i}{\hbar}(\xi x+\eta p)\right]$ is sufficient to define it for any function $A(x,p)$. This specific choice of basis mapping is the defining characteristic of the Weyl calculus.

Replacing dummy variables $q' \to q$, $p' \to p$, and $s \to z$, we obtain the inverse Weyl map (the Weyl symbol) consistent with Appendix A:

$$\boxed{A_W(q,p) = \int_{\mathbb{R}^d} d^d z \, e^{-\frac{i}{\hbar} p\cdot z} \left\langle q + \frac{z}{2} \middle| \hat{A} \middle| q - \frac{z}{2} \right\rangle.} \tag{2.3}$$

The simplest non-trivial example is the classical monomial xp. For one degree of freedom:

$$xp \mapsto \mathrm{Op}_W[xp] = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}).$$

this is called Weyl ordering. It is the ordering rule that is maximally symmetric under exchanging.

2.2 Wigner Function

At this point, we have achieved "half" of the phase-space program:

- **Observables** $\hat{A}$ can be represented as ordinary functions $A_W(q,p)$ on phase space via the Weyl symbol.

Now, we seek a phase-space representative of the state, call it $W(q,p)$, such that quantum expectation values look like classical averages.

$$\langle \hat{A} \rangle = \int\int A_W(q,p) W(q,p) \, dq dp.$$

up to a fixed normalization convention.

We begin with a pure state $|\psi\rangle$. The quantum expectation value can be written using a projector:

$$\langle \psi | \hat{A} | \psi \rangle = \mathrm{Tr}\left(|\psi\rangle\langle\psi| \hat{A} \right).$$

Naturally, we map the projector $\hat{P}_\psi = |\psi\rangle\langle\psi|$ to a phase-space function via the Weyl symbol.

Apply the Weyl symbol (Equation 2.3) to the projector:

$$W_\psi(q,p) \coloneqq \mathrm{Op}[\hat{P}_\psi](q,p) = \int_{-\infty}^{\infty} dz \, e^{-\frac{i}{\hbar} pz} \left\langle q + \frac{z}{2} \middle| \hat{P}_\psi \middle| q - \frac{z}{2} \right\rangle.$$

Since $\left\langle q + \frac{z}{2} \middle| \hat{P}_\psi \middle| q - \frac{z}{2} \right\rangle = \psi(q + \frac{z}{2})\psi^*(q - \frac{z}{2})$, we have:

$$W_\psi(q,p) = \frac{1}{(2\pi\hbar)^d} \int_{-\infty}^{\infty} dz \, e^{-\frac{i}{\hbar} pz} \psi(q + \frac{z}{2}) \psi^*(q - \frac{z}{2}).$$

For mixed states, we simply replace $\hat{P}_\psi$ by the density operator ρ. So generally, the Wigner function is

$$\boxed{W_\rho(q,p) \coloneqq \mathrm{Op}[\rho](q,p) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} d^d z \, e^{-\frac{i}{\hbar} p\cdot z} \left\langle q + \frac{z}{2} \middle| \rho \middle| q - \frac{z}{2} \right\rangle.} \tag{2.4}$$

2.3 Expectation Values in Phase Space

We now justify the central requirement posed in Section 2: once observables are represented by their Weyl symbols and states are represented by their Wigner functions, quantum expectation values take a classical-looking form.

Let ρ be a density operator and $\hat{A}$ an observable. Start from the trace written in the position basis:

$$\mathrm{Tr}\left(\rho\hat{A}\right) = \int_{-\infty}^{\infty} dq \, \langle q|\rho\hat{A}|q\rangle. \tag{2.5}$$

Insert a resolution of the identity between ρ and $\hat{A}$, $\mathbf{1} = \int dq' \, |q'\rangle\langle q'|$, to obtain

$$\mathrm{Tr}\left(\rho\hat{A}\right) = \int dq \int dq' \; \langle q|\rho|q'\rangle \, \langle q'|\hat{A}|q\rangle. \tag{2.6}$$

Introduce the midpoint and relative coordinates

$$Q := \frac{q+q'}{2}, \qquad z := q - q', \qquad \Rightarrow \qquad q = Q + \frac{z}{2}, \;\; q' = Q - \frac{z}{2}, \tag{2.7}$$

for which $dq \, dq' = dQ \, dz$. Then (2.6) becomes

$$\mathrm{Tr}\left(\rho\hat{A}\right) = \int dQ \int dz \; \left\langle Q + \frac{z}{2}\middle| \rho \middle| Q - \frac{z}{2}\right\rangle \left\langle Q - \frac{z}{2}\middle| \hat{A} \middle| Q + \frac{z}{2}\right\rangle. \tag{2.8}$$

To separate the two kernels into a product of phase-space functions, we first insert the identity

$$1 = \int dz' \, \delta(z - z')$$

into (2.8), and then use the Fourier representation of the delta distribution (from Appendix A):

$$\delta(z - z') = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp \;\; \exp\left(\frac{i}{\hbar} p(z - z')\right). \tag{2.9}$$

Substituting this and rearranging the order of integration yields

$$\mathrm{Tr}\left(\rho\hat{A}\right) = \frac{1}{2\pi\hbar} \int dQ \int dp \;\; \left[\int dz \, e^{-\frac{i}{\hbar}pz} \left\langle Q + \frac{z}{2}\middle| \rho \middle| Q - \frac{z}{2}\right\rangle\right] \left[\int dz' \, e^{\frac{i}{\hbar}pz'} \left\langle Q - \frac{z'}{2}\middle| \hat{A} \middle| Q + \frac{z'}{2}\right\rangle\right]. \tag{2.10}$$

Substituting definitions from (Equation 2.3 and Equation 2.4) into (2.10), the prefactor cancels:

$$\begin{aligned} \mathrm{Tr}\left(\rho\hat{A}\right) &= \frac{1}{2\pi\hbar} \int dQ \int dp \;\; \left[(2\pi\hbar) W_\rho(Q,p)\right] \; A_W(Q,p) \\ &= \int dQ \int dp \; W_\rho(Q,p) \, A_W(Q,p). \end{aligned} \tag{2.11}$$

Thus we obtain the phase-space pairing formula

$$\boxed{\mathrm{Tr}\left(\rho\hat{A}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_\rho(q,p) \, A_W(q,p) \, dq \, dp.} \tag{2.12}$$

2.4 Negativity of the Wigner Function

Now we can answer the most common worry:

- In classical mechanics, a state $f(q,p)$ is a nonnegative probability density.
- In quantum mechanics, $W_\rho(q,p)$ **can become negative**.

This does not contradict anything, because W_ρ is not claiming "the system has definite q and p." It simply claims that W_ρ is a phase-space representation of the state that reproduces correct measurement statistics and expectation values through phase-space integrals.

To see *how* it becomes negative, consider the first excited state of the Quantum Harmonic Oscillator, $|n = 1\rangle$. The wavefunction is proportional to $xe^{-x^2/2}$. The corresponding Wigner function is found to be:

$$W_{n=1}(q,p) = \frac{1}{\pi\hbar}\left(\frac{4}{\hbar\omega} H(q,p) - 1\right) e^{-\frac{2}{\hbar\omega}H(q,p)}, \tag{2.13}$$

where $H(q,p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$ is the classical Hamiltonian.

If we look strictly at the origin of phase space ($q = 0, p = 0$), the energy H is zero. The expression becomes:

$$W_{n=1}(0,0) = \frac{1}{\pi\hbar}(0-1)e^0 = -\frac{1}{\pi\hbar}. \tag{2.14}$$

Since $W_{n=1}(0,0) < 0$, the Wigner function cannot be interpreted as a literal probability density. These negative regions are often interpreted as signatures of non-classical "quantumness" or interference.

2.5 Properties of the Wigner Function

The definition of the Wigner function is motivated by the trace-pairing requirement, but it should also pass several immediate "sanity checks" that make it a legitimate phase-space representative of a quantum state. Throughout this subsection we assume $\rho = \rho^\dagger$ and $\mathrm{Tr}(\rho) = 1$.

(i) Reality. Starting from the definition

$$W_\rho(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dz\, e^{-\frac{i}{\hbar}pz} \left\langle x + \frac{z}{2}\middle| \rho \middle| x - \frac{z}{2}\right\rangle, \tag{2.15}$$

take the complex conjugate:

$$\begin{aligned}
W_\rho(x,p)^* &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dz\, e^{+\frac{i}{\hbar}pz} \left\langle x + \frac{z}{2}\middle| \rho \middle| x - \frac{z}{2}\right\rangle^* \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dz\, e^{+\frac{i}{\hbar}pz} \left\langle x - \frac{z}{2}\middle| \rho^\dagger \middle| x + \frac{z}{2}\right\rangle \qquad (\text{since } \langle a|\rho|b\rangle^* = \langle b|\rho^\dagger|a\rangle) \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dz\, e^{+\frac{i}{\hbar}pz} \left\langle x - \frac{z}{2}\middle| \rho \middle| x + \frac{z}{2}\right\rangle.
\end{aligned} \tag{2.16}$$

Now change variables $z \mapsto -z$ (the integration limits are unchanged):

$$W_\rho(x,p)^* = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dz\, e^{-\frac{i}{\hbar}pz} \left\langle x + \frac{z}{2}\middle| \rho \middle| x - \frac{z}{2}\right\rangle = W_\rho(x,p). \tag{2.17}$$

Hence $W_\rho(x,p) \in \mathbb{R}$ for Hermitian ρ.

(ii) Normalization. Integrating (2.15) over p produces a delta function:

$$\begin{aligned}
\int_{-\infty}^{\infty} dp\, W_\rho(x,p) &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dz\, \left\langle x + \frac{z}{2}\middle| \rho \middle| x - \frac{z}{2}\right\rangle \int_{-\infty}^{\infty} dp\, e^{-\frac{i}{\hbar}pz} \\
&= \int_{-\infty}^{\infty} dz\, \left\langle x + \frac{z}{2}\middle| \rho \middle| x - \frac{z}{2}\right\rangle \delta(z) \\
&= \langle x|\rho|x\rangle.
\end{aligned} \tag{2.18}$$

Now integrate over x:

$$\iint dx\, dp\, W_\rho(x,p) = \int dx\, \langle x|\rho|x\rangle = \mathrm{Tr}(\rho) = 1. \tag{2.19}$$

(iii) Marginals. Equation (2.18) already shows the position marginal:

$$\int_{-\infty}^{\infty} dp\, W_\rho(x,p) = \langle x|\rho|x\rangle. \tag{2.20}$$

Similarly, one obtains the momentum marginal

$$\int_{-\infty}^{\infty} dx\, W_\rho(x,p) = \langle p|\rho|p\rangle, \tag{2.21}$$

either by repeating the same calculation in the momentum representation or by applying Fourier transform identities to (2.15). Thus W_ρ reproduces the correct probability densities for x and p individually, even though it is not, in general, a nonnegative joint probability density on phase space.

2.6 The Weyl–Wigner Correspondence

$$
\begin{tikzcd}
\hat{A} \arrow[r, "\text{Weyl symbol}"] \arrow[dd, bend right] & A_W(q,p) \arrow[dd, bend left] \\
\rho \arrow[r, "\text{Wigner map}"] \arrow[d] & W_\rho(q,p) \arrow[d] \\
\mathrm{Tr}\left(\rho\,\hat{A}\right) \arrow[r, leftrightarrow, dashed, "\text{pairing identity}"] & \int_{\mathbb{R}^{2d}} W_\rho\, A_W\, \mathrm{d}q\mathrm{d}p
\end{tikzcd}
$$

Chapter 2 built the dictionary that lets us speak quantum mechanics in the language of phase space. The aim was to construct a representation in which *quantum expectation values look like classical phase-space averages* while remaining fully equivalent to the operator formulation. We have shown the following:

1. **(Observable $\leftrightarrow$ function)** We constructed the Weyl symbol map

$$\hat{A} \longmapsto A_W(q,p),$$

   resolving the ordering ambiguity by adopting Weyl (fully symmetrized) ordering. This gives a systematic way to translate operators into phase-space functions.

2. **(State $\leftrightarrow$ quasi-distribution)** We represented quantum states by the Wigner function,

$$\rho \longmapsto W_\rho(q,p),$$

   defined as the Weyl symbol of ρ.

3. **(Trace $\to$ integral with no prefactor)** Most importantly, with the normalization in Appendix A we obtained the classical-looking pairing rule

$$\mathrm{Tr}\left(\rho\hat{A}\right) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W_\rho(q,p)\, A_W(q,p)\, \mathrm{d}^d q\, \mathrm{d}^d p,$$

4. **(Interpretation)** We clarified why W_ρ is a *quasi*-probability: it is real, normalized, and has correct marginals, but need not be nonnegative. This is the minimal price for representing states on phase space without contradicting uncertainty.

3 The Moyal ⋆-Product

"An attempt is made to interpret quantum mechanics as a statistical theory, or more exactly as a form of non-deterministic statistical dynamics."

– J.E. Moyal, *Quantum mechanics as a statistical theory* (1949)

Chapter 2 built the translation layer between the operator formulation of quantum mechanics and a phase-space formulation: observables $\hat{A}$ were sent to Weyl symbols $A_W(q,p)$, states ρ were sent to Wigner functions $W_\rho(q,p)$, and trace pairing became an ordinary phase-space integral,

$$\mathrm{Tr}\Big(\rho\hat{A}\Big) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W_\rho(q,p)\, A_W(q,p)\, d^d q\, d^d p.$$

At first sight this looks like the end of the story: classical statistical mechanics is "a state times an observable, integrated over phase space," and we have obtained exactly that form. It is therefore natural to suspect that quantum mechanics on phase space is simply classical mechanics with an unusual (quasi-)distribution.

This suspicion is the *trap*. In classical mechanics, the algebra of observables is built from the pointwise product:

$$(AB)(q,p) = A(q,p)\, B(q,p).$$

If the Weyl–Wigner correspondence behaved like an ordinary change of variables, one might expect the same rule to hold after quantization, namely

$$(\hat{A}\hat{B})_W(q,p) \stackrel{?}{=} A_W(q,p)\, B_W(q,p).$$

But this is not true in general. The reason is structural: operator multiplication in Hilbert space is noncommutative, whereas pointwise multiplication of functions is commutative. Even before we compute anything, this mismatch already signals that the phase-space representation cannot preserve the product structure without modification. In other words, Chapter 2 gave us an isomorphism of vector spaces (operators ↔ symbols), but not an isomorphism of algebras.

The objective of this chapter is to repair exactly this missing piece. We will introduce a new, $\hbar$-dependent product $\star$ on phase-space functions-the *Moyal star product*, defined such that:

$$(\hat{A}\hat{B})_W \;=\; A_W \star B_W.$$

This single replacement completes the phase-space formulation of quantum mechanics: once pointwise multiplication is deformed into ⋆-multiplication, commutators become ⋆-commutators, and the quantum equations of motion become phase-space evolution equations generated by the Moyal bracket. Moreover, the deformation is controlled: in the classical limit $\hbar \to 0$, $\star$ reduces to the usual pointwise product, and the Moyal bracket reduces to the Poisson bracket. Thus the "deformation" captures, in a precise algebraic sense, how classical mechanics emerges as the leading approximation to quantum mechanics.

3.1 Constructing the ⋆-product

Our goal is to compute the Weyl symbol of a product $\hat{C} = \hat{A}\hat{B}$ directly from the symbols A_W and B_W. The strategy is a "translate–multiply–translate back" pipeline[2]:

1. **Translate** each operator into its *position-space kernel* $K_{\hat{A}}(x,y) = \langle x|\hat{A}|y\rangle$, because in the Schrödinger representation operator multiplication becomes the familiar integral composition of kernels (matrix multiplication).

2. **Multiply** using that kernel rule to obtain $K_{\hat{C}}$.

3. **Translate back** by applying the inverse Weyl map to $K_{\hat{C}}$, which produces a closed formula $C_W = A_W \star B_W$.

[2]The derivation presented here is primarily adapted from [6], with extra explanations added for clarity.

Working with Kernels

At first sight it may seem odd to switch from symbols back to kernels. The reason is that the product $\hat{A}\hat{B}$ is simplest in a basis. In the position basis, operators act as integral transforms

$$\begin{aligned}(\hat{A}\psi)(x) \equiv \langle x| \hat{A} |\psi\rangle &= \langle x| \hat{A} \int_{\mathbb{R}^d} d^d y \; |y\rangle\langle y| |\psi\rangle \\ &= \int_{\mathbb{R}^d} d^d y \; \langle x| \hat{A} |y\rangle \langle y|\psi\rangle .\end{aligned}$$

And by the definition of the kernel

$$(\hat{A}\psi)(x) = \int_{\mathbb{R}^d} K_{\hat{A}}(x,y)\, \psi(y)\, d^d y, \qquad K_{\hat{A}}(x,y) := \langle x|\hat{A}|y\rangle,$$

and operator multiplication becomes the elementary matrix rule

$$\boxed{K_{\hat{A}\hat{B}}(x,y) = \int_{\mathbb{R}^d} d^d z\, K_{\hat{A}}(x,z)\, K_{\hat{B}}(z,y).} \tag{3.1}$$

This is the continuum analogue of multiplying matrices ($(AB)_{ij} = \sum_k A_{ik}B_{kj}$). In contrast, trying to multiply symbols directly has no obvious rule (that is precisely the problem we are solving).

Why introduce a $2d$-dimensional Fourier transform of a symbol?

The kernel formulas that connect symbols and operators involve oscillatory factors $e^{\frac{i}{\hbar}p\cdot(x-y)}$ and midpoint coordinates $\frac{x+y}{2}$. When we substitute these into the product rule (3.1), we encounter integrals over several variables with phase factors that are *linear* in those variables. The standard way to evaluate such integrals is to Fourier-expand the remaining dependence so that the p-integrals produce delta distributions.

The $2d$-dimensional Fourier transform exploits the fact that the symbol is a function on phase space $\mathbb{R}^{2d}$. Any sufficiently well-behaved function on $\mathbb{R}^{2d}$ admits a Fourier representation in its $2d$ variables, so we may write

$$A_W(q,p) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} d^d\alpha\, d^d\beta\; \widetilde{A}(\alpha,\beta)\; \exp\left(\frac{i}{\hbar}(\alpha\cdot p + \beta\cdot q)\right), \tag{3.2}$$

with inverse

$$\widetilde{A}(\alpha,\beta) = \frac{1}{(2\pi\hbar)^{2d}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} d^d p\, d^d q\; A_W(q,p)\; \exp\left(-\frac{i}{\hbar}(\alpha\cdot p + \beta\cdot q)\right). \tag{3.3}$$

The payoff of (3.2) is that multiplication by α, β and differentiation with respect to q, p become interchangeable (Fourier duality). This is exactly the mechanism that will produce the final bidifferential operator in the $\star$-product.

Kernel $\leftrightarrow$ symbol in a convenient Fourier form

From Weyl quantisation the kernel can be written in terms of the symbol as

$$K_{\hat{A}}(x,y) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} A\left(\frac{x+y}{2}, p\right) \exp\left(\frac{i}{\hbar}p\cdot(x-y)\right)\, d^d p. \tag{3.4}$$

Now insert the Fourier representation (3.2) of A into (3.4):

$$K_{\hat{A}}(x,y) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} d^d p\; A\left(\frac{x+y}{2}, p\right)\; e^{\frac{i}{\hbar}p\cdot(x-y)}$$

$$= \frac{1}{(2\pi\hbar)^{2d}} \int_{\mathbb{R}^d} d^d p \int_{\mathbb{R}^d} d^d\alpha\, d^d\beta\, \widetilde{A}(\alpha,\beta)\, \exp\left(\frac{i}{\hbar}\left[p\cdot\alpha + \frac{x+y}{2}\cdot\beta\right]\right) \exp\left(\frac{i}{\hbar} p\cdot(x-y)\right)$$

$$= \frac{1}{(2\pi\hbar)^{2d}} \int_{\mathbb{R}^d} d^d\alpha\, d^d\beta\, \widetilde{A}(\alpha,\beta)\, \exp\left(\frac{i}{\hbar}\frac{x+y}{2}\cdot\beta\right) \int_{\mathbb{R}^d} d^d p\, \exp\left(\frac{i}{\hbar} p\cdot(x-y+\alpha)\right). \tag{3.5}$$

Using

$$\int_{\mathbb{R}^d} d^d p\, \exp\left(\frac{i}{\hbar} p\cdot u\right) = (2\pi\hbar)^d\, \delta(u),$$

the p-integral gives $(2\pi\hbar)^d \delta(x-y+\alpha)$, hence

$$\boxed{K_{\hat{A}}(x,y) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} \widetilde{A}(y-x,\beta)\, \exp\left(\frac{i}{\hbar}\frac{x+y}{2}\cdot\beta\right) d^d\beta.} \tag{3.6}$$

Symbol of The Product

Start from the inverse Weyl map:

$$C(q,p) = \mathrm{Op}(\hat{A}\hat{B})(q,p) = \int_{\mathbb{R}^d} K_{\hat{A}\hat{B}}\left(q+\frac{z}{2}, q-\frac{z}{2}\right) \exp\left(-\frac{i}{\hbar} p\cdot z\right) d^d z. \tag{3.7}$$

Insert the kernel product rule (3.1):

$$C(q,p) = \int_{\mathbb{R}^d} d^d z \int_{\mathbb{R}^d} d^d y\, K_{\hat{A}}\left(q+\frac{z}{2}, y\right) K_{\hat{B}}\left(y, q-\frac{z}{2}\right) \exp\left(-\frac{i}{\hbar} p\cdot z\right). \tag{3.8}$$

Now substitute the Fourier-kernel form (3.6) for each factor:

$$\begin{aligned} C(q,p) = \int d^d z \int d^d y \int d^d\alpha \int d^d\beta\, \widetilde{A}\left(y-q-\frac{z}{2}, \alpha\right) \exp\left(\frac{i}{\hbar}\alpha\cdot\frac{q+\frac{z}{2}+y}{2}\right) \\ \times \widetilde{B}\left(q-\frac{z}{2}-y, \beta\right) \exp\left(\frac{i}{\hbar}\beta\cdot\frac{y+q-\frac{z}{2}}{2}\right) \exp\left(-\frac{i}{\hbar} p\cdot z\right). \end{aligned} \tag{3.9}$$

Symmetrizing Change of Variables

Introduce the variables

$$X := y - q - \frac{z}{2}, \qquad Y := q - \frac{z}{2} - y. \tag{3.10}$$

Then one checks immediately that

$$z = -(X+Y), \qquad y = q + \frac{X-Y}{2}. \tag{3.11}$$

Substituting (3.10)–(3.1) into (3.9) (and relabelling dummy variables so that the arguments of $\widetilde{A}$ and $\widetilde{B}$ are simply X and Y) yields the compact expression

$$\boxed{C(p,q) = \int d^d X\, d^d Y\, d^d\alpha\, d^d\beta\, \widetilde{A}(X,\alpha)\, \widetilde{B}(Y,\beta)\, \exp\left(\frac{i}{\hbar}\left[q\cdot(\alpha+\beta) + p\cdot(X+Y)\right]\right) \exp\left(\frac{i}{2\hbar}(\beta\cdot X - \alpha\cdot Y)\right).} \tag{3.12}$$

The first exponential reconstructs $A(p,q)$ and $B(p,q)$ from their Fourier transforms, while the second exponential is the *extra phase* (the "twist") responsible for quantum corrections. In the sense that, if the second exponential equals to 1, we can break the first exponential into two independent blocks:

$$C(q,p) = \underbrace{\left[\iint dX d\alpha \tilde{A}(X,\alpha) e^{\frac{i}{\hbar}(q\cdot\alpha + p\cdot X)}\right]}_{\text{Inverse Fourier Transform of } \hat{A}} \times \underbrace{\left[\iint dY d\beta \tilde{B}(Y,\beta) e^{\frac{i}{\hbar}(q\cdot\beta + p\cdot Y)}\right]}_{\text{Inverse Fourier Transform of } \hat{B}} \times \underbrace{1}_{\text{Extra Phase}}.$$

Notice the problematic extra-phase term:

$$\exp\left\{\frac{i}{2\hbar}(\beta\cdot X-\alpha\cdot Y)\right\}.$$

It takes β (momentum-like variable from B) and multiplies it by X (position-like variable from A). It takes α (momentum-like variable from A) and multiplies it by Y (the position-like variable from B). It couples the integrals. Values of $\tilde{A}(X,\alpha)$ at one set of Fourier variables are weighted by values of $\tilde{B}(Y,\beta)$ at another, and the result at (q,p) is obtained only after *integrating over all* X, Y, α, β. Since $\tilde{A}$ and $\tilde{B}$ themselves are Fourier transforms of the kernels (which in turn are built from the wavefunction/density matrix), this coupling means that $(A\star B)(q,p)$ depends on the full global shape of the underlying kernel.

Integral → Differential Expression

Expand the twist exponential in a power series:

$$\exp\left(\frac{i}{2\hbar}(\beta\cdot X-\alpha\cdot Y)\right)=\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{i}{2\hbar}(\beta\cdot X-\alpha\cdot Y)\right]^{n}. \tag{3.13}$$

Now use Fourier duality: inside the integral (3.12),

$$X_j\exp\left(\frac{i}{\hbar}p\cdot X\right)=\frac{\hbar}{i}\frac{\partial}{\partial p_j}\exp\left(\frac{i}{\hbar}p\cdot X\right),\qquad \alpha_j\exp\left(\frac{i}{\hbar}q\cdot\alpha\right)=\frac{\hbar}{i}\frac{\partial}{\partial q_j}\exp\left(\frac{i}{\hbar}q\cdot\alpha\right),$$

and similarly for Y and β. Note that

> X and α appear only in the Fourier factor associated with $\widetilde{A}$, hence the resulting derivatives act only on A; likewise Y and β produce derivatives acting only on B.

This is why the final expression uses left- and right-acting derivatives. Carrying this out term-by-term in (3.13) and then performing the (X,α)-integrals and (Y,β)-integrals (which reconstruct $A(p,q)$ and $B(p,q)$ via (3.2)) yields (See Appendix C for the full derivation.)

$$C(p,q)=\sum_{n=0}^{\infty}\frac{1}{n!}\,A(p,q)\left[\frac{\hbar}{2i}\left(\overleftarrow{\nabla}_p\cdot\overrightarrow{\nabla}_q-\overleftarrow{\nabla}_q\cdot\overrightarrow{\nabla}_p\right)\right]^{n}B(p,q). \tag{3.14}$$

The Moyal product and the classical limit

Define the $\star$-product by $C=\mathrm{Op}(\hat{A}\hat{B})=A\star B$. Then (3.14) can be summed into the exponential form

$$\boxed{(A\star B)(p,q)=A(p,q)\,\exp\left[\frac{\hbar}{2i}\left(\overleftarrow{\nabla}_p\cdot\overrightarrow{\nabla}_q-\overleftarrow{\nabla}_q\cdot\overrightarrow{\nabla}_p\right)\right]B(p,q).} \tag{3.15}$$

Expanding the exponential shows immediately that the leading term is pointwise multiplication and the first correction contains the Poisson bracket:

$$\begin{aligned}A\star B&=AB+\frac{\hbar}{2i}\left(\nabla_pA\cdot\nabla_qB-\nabla_qA\cdot\nabla_pB\right)+O(\hbar^2)\\&=AB+\frac{i\hbar}{2}\{A,B\}+O(\hbar^2),\qquad \{A,B\}:=\nabla_qA\cdot\nabla_pB-\nabla_pA\cdot\nabla_qB.\end{aligned} \tag{3.16}$$

Thus $\star$ is literally a *deformation* of the classical product: as $\hbar\to 0$,

$$A\star B\longrightarrow AB,$$

but at finite $\hbar$ the extra derivative terms encode the noncommutative operator algebra in purely phase-space language.

3.2 The Moyal bracket and phase-space dynamics

So far, the $\star$-product has been motivated algebraically: it is the unique replacement for pointwise multiplication that makes the Weyl–Wigner correspondence compatible with operator composition,

$$(\hat{A}\hat{B})_W = A_W \star B_W.$$

But quantum mechanics is not just an algebra of observables, it is also a theory of time evolution. In the operator (Heisenberg) picture, the dynamics of an observable $\hat{A}(t)$ is governed by the commutator with the Hamiltonian $\hat{H}$:

$$\frac{d}{dt}\hat{A}(t) = \frac{1}{i\hbar}\,[\hat{A}(t), \hat{H}] \qquad \text{(Heisenberg equation).} \tag{3.17}$$

If the phase-space formulation is to be more than a change of notation, it must be able to translate (3.17) into an equation purely on phase space.

Moyal bracket

Because the Weyl map turns operator products into $\star$-products, it turns commutators into $\star$-commutators. This motivates the *Moyal bracket*:

$$\boxed{\{A, B\}_M := \frac{1}{i\hbar}\big(A \star B - B \star A\big).} \tag{3.18}$$

Formally, $\{\cdot,\cdot\}_M$ plays the same role in deformation quantization that the Poisson bracket $\{\cdot,\cdot\}$ plays in classical mechanics, which is generating time evolution.

Phase-space Equation of Motion

Let $A_W(q,p,t)$ be the Weyl symbol of the Heisenberg operator $\hat{A}(t)$, and let $H_W(q,p)$ be the Weyl symbol of the Hamiltonian $\hat{H}$.[3] Taking Weyl symbols on both sides of (3.17) and using $(\hat{A}\hat{H})_W = A_W \star H_W$ and $(\hat{H}\hat{A})_W = H_W \star A_W$ gives the phase-space dynamics:

$$\boxed{\frac{d}{dt}A_W(q,p,t) = \{A_W(q,p,t),\ H_W(q,p)\}_M.} \tag{3.19}$$

Equivalently, for a density operator $\rho(t)$ one can derive a phase-space version of the von Neumann equation, giving an evolution equation for the Wigner function $W_\rho(q,p,t)$. The point is the same: quantum time evolution is classical Hamiltonian flow, but with the Poisson bracket replaced by the Moyal bracket.

Classical limit: why the first correction is $O(\hbar^2)$ (not $O(\hbar)$)

To connect (3.19) to classical mechanics, expand the $\star$-product in powers of $\hbar$. In one dimension (for notational simplicity),

$$A \star B = AB + \frac{i\hbar}{2}\{A,B\} - \frac{\hbar^2}{8}\Big(\partial_q^2 A\,\partial_p^2 B - 2\,\partial_q\partial_p A\,\partial_p\partial_q B + \partial_p^2 A\,\partial_q^2 B\Big) + O(\hbar^3), \tag{3.20}$$

where $\{A,B\} = \partial_q A\,\partial_p B - \partial_p A\,\partial_q B$ is the classical Poisson bracket.

Now form the $\star$-commutator:

$$A \star B - B \star A.$$

The key observation is *antisymmetry*. The AB term cancels immediately because ordinary multiplication is commutative: $AB - BA = 0$. More subtly, the *even* powers of $\hbar$ cancel as well: the $\hbar^2$ term in (3.20)

[3] For time-independent Hamiltonians this is the most common setting; time-dependent cases can be treated similarly.

is *symmetric* under exchanging A and B (it is built from products of second derivatives), so it subtracts to zero when we take $A \star B - B \star A$. In general, the $\star$-product has the structure

$$A \star B = AB + (i\hbar)\,(\text{antisymmetric piece}) + \hbar^2(\text{symmetric piece}) + (i\hbar)^3(\text{antisymmetric piece}) + \cdots, \tag{3.21}$$

so the commutator keeps only the *odd* powers.

Dividing by $i\hbar$ in the definition (3.18) then yields

$$\boxed{\{A, B\}_M = \{A, B\} + O(\hbar^2).} \tag{3.22}$$

This is an important point for interpretation: the Moyal bracket reproduces the Poisson bracket at leading order, and the first quantum correction to classical Hamiltonian flow appears only at order $\hbar^2$ (not $\hbar$). The antisymmetry kills all terms in the $\star$-expansion that are symmetric under exchanging A and B.

With the definition (3.18) and the evolution law (3.19) in place, we are ready to compute concrete examples. The next subsection shows that $\star$ immediately reproduces the canonical commutation relation, and that for quadratic Hamiltonians the Moyal bracket coincides exactly with the Poisson bracket.

3.3 Worked Examples

The star product is defined by an exponential of derivatives, which can look abstract at first sight. This subsection gives two short computations that can be done by hand.

3.3.1 Example 1: The Canonical $\star$-Commutator

Let q and p denote the *coordinate functions* on phase space. Using the bidifferential form

$$(A \star B)(q, p) = A(q, p) \exp\left[\frac{i\hbar}{2}\left(\overleftarrow{\partial_q}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_q}\right)\right] B(q, p), \tag{3.23}$$

we compute $q \star p$ and $p \star q$. Since q and p are linear functions, all second and higher derivatives vanish, so the exponential truncates after the first correction term:

$$A \star B = AB + \frac{i\hbar}{2}\left(\partial_q A\, \partial_p B - \partial_p A\, \partial_q B\right). \tag{3.24}$$

Take $A = q$ and $B = p$. Then $\partial_q q = 1$, $\partial_p q = 0$, $\partial_q p = 0$, and $\partial_p p = 1$, hence

$$q \star p = qp + \frac{i\hbar}{2}(1 \cdot 1 - 0 \cdot 0) = qp + \frac{i\hbar}{2}. \tag{3.25}$$

Similarly, for $A = p$ and $B = q$ we have $\partial_q p = 0$, $\partial_p p = 1$, $\partial_q q = 1$, $\partial_p q = 0$, giving

$$p \star q = pq + \frac{i\hbar}{2}(0 \cdot 0 - 1 \cdot 1) = pq - \frac{i\hbar}{2}. \tag{3.26}$$

Subtracting,

$$\boxed{q \star p - p \star q = i\hbar.} \tag{3.27}$$

Thus the $\star$-commutator reproduces the canonical operator commutator:

$$[\hat{q}, \hat{p}] = i\hbar \quad \Longleftrightarrow \quad [q, p]_\star := q \star p - p \star q = i\hbar.$$

3.3.2 Example 2: Quadratic Hamiltonians

A major simplification occurs for Hamiltonians that are at most quadratic in (q, p). Consider the one-dimensional harmonic oscillator Weyl Hamiltonian

$$H(q, p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2. \tag{3.28}$$

We claim that for any smooth phase-space function $A(q, p)$,

$$\boxed{\{A, H\}_M = \{A, H\},} \tag{3.29}$$

i.e. the Moyal bracket coincides *exactly* with the Poisson bracket.
To see why, recall the series expansion of the Moyal bracket (obtained by expanding the star product in powers of $\hbar$):

$$\{A, H\}_M = \{A, H\} + \frac{\hbar^2}{24}\left(\partial_q^3 A\, \partial_p^3 H - 3\, \partial_q^2 \partial_p A\, \partial_p^2 \partial_q H + 3\, \partial_q \partial_p^2 A\, \partial_p \partial_q^2 H - \partial_p^3 A\, \partial_q^3 H\right) + O(\hbar^4). \tag{3.30}$$

The pattern is the key point: beyond the Poisson bracket, all corrections involve *third and higher* derivatives of H (and of A). If H is quadratic, then

$$\partial_q^3 H = 0, \qquad \partial_p^3 H = 0, \qquad \text{and in fact all derivatives of } H \text{ of order} \geqslant 3 \text{ vanish.}$$

Therefore every correction term in (3.30) is identically zero, leaving only the Poisson bracket. In the harmonic oscillator case (3.28), we thus obtain

$$\frac{d}{dt}A(q, p, t) = \{A, H\}_M = \{A, H\},$$

so the phase-space dynamics is *exactly classical* in form. The quantum behaviour of the oscillator is then encoded not in modified trajectories, but in the fact that the allowed states (Wigner functions) can have interference structures and negative regions, even though the flow in phase space is generated by the classical Hamiltonian vector field.

As a concrete check, take $A(q, p) = q$ and $A(q, p) = p$. Using the Poisson bracket with H in (3.28),

$$\dot{q} = \{q, H\} = \partial_p H = \frac{p}{m}, \qquad \dot{p} = \{p, H\} = -\partial_q H = -m\omega^2 q,$$

which are precisely Hamilton's equations. By (3.29), these are also the Moyal equations of motion for q and p.

4 Quantum-Classical Transition in the Wigner Picture

"The quantum description of the microscopic world is incompatible with the classical description of the macroscopic world, both mathematically and conceptually. Nevertheless, it is generally accepted that classical mechanics emerges from quantum mechanics in the macroscopic limit."
– R. Ramanathan, T. Paterek, et al., *Local Realism of Macroscopic Correlations*

By this point we have transported quantum mechanics onto phase space: observables are Weyl symbols $A_W(q,p)$, states are Wigner functions $W_\rho(q,p)$, and operator composition is encoded by the Moyal $\star$-product. Although the formal limit $\hbar \to 0$ makes $\star$ reduce to pointwise multiplication, this is not the way classicality appears operationally. Real measurements have finite resolution and cannot access arbitrarily fine phase-space structure. In this chapter we therefore treat the quantum–classical transition as an *information-loss problem.* We first summarize how macroscopic (collective) measurements can admit an effectively classical description [8], and then model this restricted access in the Wigner picture as a coarse-graining (smoothing) map on phase space.

4.1 Macroscopic Correlations and Effective Local Realism

Consider a large composite system consisting of an enormous number of microscopic constituents, $N \gg 1$ (in a literal macroscopic sample one may have $N \sim 10^{23}$). Two distant observers, Alice and Bob, each have access to a large subsystem. The essential restriction is that their measurement apparatus does not address individual constituents. Instead, it outputs only **collective** or **spatially averaged** quantities, observables that explicitly average over many particles.

A canonical example is collective magnetization in a many-spin sample. For measurement settings labelled by unit vectors $\mathbf{a}$ and $\mathbf{b}$, one may consider collective observables of the form

$$M_A(\mathbf{a}) \;\propto\; \sum_{i=1}^{N_A} \sigma_{\mathbf{a}}^{(i)}, \qquad M_B(\mathbf{b}) \;\propto\; \sum_{j=1}^{N_B} \sigma_{\mathbf{b}}^{(j)}, \tag{4.1}$$

where $\sigma_{\mathbf{a}}^{(i)}$ is the Pauli spin of the ith constituent along direction $\mathbf{a}$, and similarly for Bob. In optical language one may instead think of total intensity, coarse photon-number counts, or other extensive observables built as sums over many modes.

Crucially, macroscopic detection is typically *binned* or *coarse.* Many distinct microscopic configurations are mapped to the same macroscopic readout. Operationally, this means that Alice and Bob's data is a averaged description of the microscopic state. Even if the underlying many-body state is entangled at the microscopic level, the measurement returns only a collective summary.

In microscopic Bell tests, one probes correlations between outcomes of well-controlled local measurements and asks whether the observed statistics admit a classical explanation in terms of a **local hidden-variable** (LHV) model. Ramanathan et al. [8] pose the analogous question at the macroscopic level:

> If Alice and Bob are restricted to macroscopic observables such as (4.1), do the resulting correlations still violate local realism (Bell inequalities), or do they reduce to classical behaviour?

The point is that the *accessible* statistics may fail to retain the nonclassical information.

Their conclusion can be summarized conceptually as follows. Even when the microscopic constituents are described by quantum mechanics and may be entangled, the correlations observable through *macroscopic* measurements can admit an effective classical (LHV) description.

The interpretation relevant for us is that **classicality can emerge as an operational property of restricted access**. The microscopic state need not become classical; rather, the macroscopic interface to that state (coarse measurement) removes precisely the features that would certify nonclassicality.

4.2 Coarse-Graining and the Operational Classical Limit

We now translate "macroscopic access" into phase-space language. The central principle is the Weyl–Wigner pairing established earlier:

$$\mathrm{Tr}\left(\rho\hat{A}\right) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W_\rho(q,p)\, A_W(q,p)\, d^d q\, d^d p, \tag{4.2}$$

so predictions depend only on the phase-space product $W_\rho A_W$ under integration. This lets us model limited resolution as an *information-loss map* acting on phase-space functions.

4.2.1 Coarse-graining the State and the Observable

Let $\mathcal{C}_\sigma$ denote a coarse-graining map implemented by convolution with a normalized kernel G_σ of width σ:

$$(\mathcal{C}_\sigma F)(q,p) := (G_\sigma * F)(q,p) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G_\sigma(q-q', p-p')\, F(q',p')\, d^d q'\, d^d p', \qquad \int G_\sigma = 1. \tag{4.3}$$

Since only (4.2) matters, we have two equivalent ways to implement this limited resolution.

(A) Coarse-grained state. Define the **coarse-grained Wigner function** by

$$W_\rho^{(\sigma)} := \mathcal{C}_\sigma W_\rho = G_\sigma * W_\rho. \tag{4.4}$$

This is not the claim that the microscopic state changes; it is the effective phase-space portrait accessible under finite resolution.

(B) Coarse-grained observable. Alternatively, define the **coarse-grained Weyl symbol** by

$$A_W^{(\sigma)} := \mathcal{C}_\sigma A_W = G_\sigma * A_W. \tag{4.5}$$

This reflects that a real apparatus cannot implement an infinitely sharp phase-space probe.

These pictures are interchangeable because convolution is self-adjoint with respect to the L^2 pairing. For symmetric kernels $G_\sigma(q,p) = G_\sigma(-q,-p)$ (in particular for a Gaussian),

$$\int W_\rho^{(\sigma)}(q,p)\, A_W(q,p)\, d^d q\, d^d p = \int W_\rho(q,p)\, A_W^{(\sigma)}(q,p)\, d^d q\, d^d p. \tag{4.6}$$

Hence macroscopic predictions may be written in either form:

$$\mathrm{Tr}\left(\rho\hat{A}\right) = \int W_\rho^{(\sigma)}\, A_W = \int W_\rho\, A_W^{(\sigma)}. \tag{4.7}$$

Mathematically, one may attach the information loss to the "state side" or the "measurement side" without changing the predicted expectation value.

4.2.2 Coarse-graining as a Low-Pass Filter

To see what coarse-graining removes, we switch to Fourier space. Define the phase-space Fourier transform (one degree of freedom for simplicity) by [4]

$$\widetilde{F}(k_q, k_p) := \int_{\mathbb{R}}\int_{\mathbb{R}} F(q,p)\, e^{-i(k_q q + k_p p)}\, dq\, dp. \tag{4.8}$$

[4] Here we use the engineering Fourier convention (no $\hbar$ in the exponential), k has units inverse to (q,p).

By the Convolution Theorem,

$$\widetilde{\mathcal{C}_\sigma F}(k_q, k_p) = \widetilde{G}_\sigma(k_q, k_p)\, \widetilde{F}(k_q, k_p). \tag{4.9}$$

For a Gaussian kernel (with separate resolutions σ_q, σ_p),

$$\widetilde{G}_\sigma(k_q, k_p) = \exp\left(-\frac{\sigma_q^2 |k_q|^2}{2} - \frac{\sigma_p^2 |k_p|^2}{2}\right). \tag{4.10}$$

Thus $\widetilde{G}_\sigma$ acts as a *transfer function*: it is close to 1 for small wavevectors and decays exponentially for large wavevectors. In signal-processing terms, coarse detection implements a **low-pass filter**. The cutoff occurs roughly when $\sigma k \sim 1$, i.e. when $|k_q| \gtrsim 1/\sigma_q$ or $|k_p| \gtrsim 1/\sigma_p$. Fine phase-space ripples are high frequency and are precisely what is removed.

4.2.3 Why the $\star$-Product Gets Washed Out

We now show that the same filtering mechanism suppresses the *deformation* itself. Recall the exponential derivative form of the Moyal product:

$$(A \star B)(q, p) = A(q, p)\, \exp\left[\frac{i\hbar}{2}\left(\overleftarrow{\partial_q}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_q}\right)\right]\, B(q, p), \tag{4.11}$$

so that

$$A \star B = AB + \frac{i\hbar}{2}\Big(\partial_q A\, \partial_p B - \partial_p A\, \partial_q B\Big) + \mathcal{O}(\hbar^2). \tag{4.12}$$

Every correction beyond AB involves derivatives. But differentiation corresponds to multiplication by wavevector in Fourier space:

$$\partial_q \longleftrightarrow ik_q, \qquad \partial_p \longleftrightarrow ik_p. \tag{4.13}$$

Therefore higher-order $\star$-corrections are weighted by higher powers of k_q and k_p, meaning they are most sensitive to *high-frequency* (fine-scale) phase-space structure.

After coarse-graining, the accessible bandwidth is effectively limited to

$$|k_q| \lesssim \frac{1}{\sigma_q}, \qquad |k_p| \lesssim \frac{1}{\sigma_p}, \tag{4.14}$$

since larger wavevectors are exponentially suppressed by (4.10). Therefore, the magnitude of the first quantum correction in (4.12) scales like

$$\hbar\, |k_q||k_p| \;\lesssim\; \frac{\hbar}{\sigma_q \sigma_p}.$$

More generally, the nth-order terms are suppressed by higher powers of this ratio. Hence, when the measurement resolution satisfies

$$\sigma_q \sigma_p \gg \hbar, \tag{4.15}$$

the surviving bandwidth is sufficiently low that the $\star$-product becomes indistinguishable from pointwise multiplication on the coarse-grained data:

$$\boxed{A \star B \;\approx\; AB \qquad \text{whenever the retained phase-space bandwidth is low enough.}} \tag{4.16}$$

This is the precise sense in which macroscopic resolution describes a commutative (classical-looking) algebra, since the deformation terms are effectively removed.

4.2.4 Dynamics: Moyal bracket $\to$ Poisson bracket

The same reasoning applies to time evolution. In phase space, commutators become Moyal brackets,

$$\{A, H\}_M := \frac{1}{i\hbar}\big(A \star H - H \star A\big), \tag{4.17}$$

and expanding $\star$ yields

$$\{A, H\}_M = \{A, H\} + \mathcal{O}(\hbar^2), \tag{4.18}$$

where $\{\cdot,\cdot\}$ is the classical Poisson bracket. Under coarse-graining, the higher-derivative terms responsible for the $\mathcal{O}(\hbar^2)$ corrections are suppressed by the same bandwidth filtering mechanism, so macroscopic observables evolve approximately classically.

4.3 Application: The Husimi Q-Function

The coarse-graining map $\mathcal{C}_\sigma$ was introduced as a model of finite experimental resolution. A natural question is: *is there a canonical choice of smoothing that corresponds to a physically meaningful phase-space distribution?* The answer is yes: the **Husimi Q-function**. Conceptually, the Husimi function is what you get when you measuring the phase space by a physically allowed probe with minimal uncertainty.

The Wigner function $W_\rho(q, p)$ is the correct phase-space representative of a quantum state, but it is not a genuine probability densit due to the lack of positivity.

So we ask a more physical question:

> *What phase-space distribution corresponds to the outcomes of an actually implementable measurement with finite resolution?*

4.3.1 Coherent States

A standard family of minimum-uncertainty states are **coherent states**, denoted $|\alpha\rangle$. They are Gaussian wavepackets centred at a phase-space point $\alpha \equiv (q, p)$ with uncertainties satisfying the minimum uncertainty product

$$\Delta q \, \Delta p = \frac{\hbar}{2}. \tag{4.19}$$

which makes them the closest quantum analogue of a classical point in phase space: the smallest allowable "blob".

Now imagine probing the state ρ with this minimal blob. The probability that ρ "looks like" the coherent state centred at α is exactly the overlap $\langle\alpha|\rho|\alpha\rangle$.

4.3.2 Definition and physical meaning of the Husimi Q-function

This leads directly to the Husimi function:

$$\boxed{Q_\rho(\alpha) := \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle} \qquad \text{(one degree of freedom)}. \tag{4.20}$$

Three points make (4.20) extremely useful:

1. $Q_\rho(\alpha) \geqslant 0$ **always**. It is an expectation value of a positive operator, so it cannot be negative. In this sense, Q is a genuine probability density (up to convention) [5].
2. **It has a direct measurement interpretation.** The quantity $\langle\alpha|\rho|\alpha\rangle$ is the probability density for obtaining the outcome associated with a coherent-state probe (more formally, a POVM built from coherent states).

[5] Even though it is positive semidefinite, we classify it as a quasiprobability distribution because it has no unique joint probability, hence failing to satisfy the Kolmogorov's third axiom of probability. See [10] for a discussion on this topic.

3. **It is exactly a coarse-grained Wigner function.** Q can be obtained by smoothing W_ρ with a Gaussian of minimal uncertainty width.

4.3.3 Husimi Q as Gaussian smoothing of W

The connection to the present chapter is that the Husimi function is *precisely* the Wigner function blurred at the minimal uncertainty scale. Schematically,

$$Q_\rho(q,p) = \big(G_{\min} * W_\rho\big)(q,p), \tag{4.21}$$

where $G_{\min}$ is a Gaussian kernel whose widths satisfy $\sigma_q \sigma_p = \hbar/2$. This is the sharpest smoothing that is still consistent with the uncertainty principle.

Since Q_ρ is a smoothed version of W_ρ, it suppresses the interference fringes and rapid oscillations that are responsible for Wigner negativity. In the language of Section 4.3, it also suppresses the high-bandwidth structures on which the $\star$-product deformation depends. Thus the Husimi Q-function provides a standard example of an *operational classical limit*: it is the phase-space state description appropriate to a macroscopic observer who can only probe phase space with finite resolution.

4.3.4 The Distribution of a Phase-space Measurement

A key advantage of the Husimi Q-function is that it is not merely a “nicer” (positive) plot of a quantum state: it is directly tied to a concrete measurement procedure. The central observation is that quantum mechanics does not allow a perfectly sharp, simultaneous measurement of q and p. The best one can do operationally is probe phase space with a *minimum-uncertainty wavepacket.* Coherent states $|\alpha\rangle$ provide exactly such probes: they are Gaussian packets centred at a phase-space point $\alpha \equiv (q,p)$ and satisfy $\Delta q\, \Delta p = \hbar/2$.

The corresponding measurement is described by the coherent-state POVM

$$E(\alpha) = \frac{1}{\pi}\,|\alpha\rangle\langle\alpha|, \qquad \int d^2\alpha\; E(\alpha) = \mathbf{1}, \tag{4.22}$$

so that the probability density of obtaining outcome α when measuring the state ρ is

$$p(\alpha) = \mathrm{Tr}\big(\rho\, E(\alpha)\big) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle. \tag{4.23}$$

But the right-hand side is precisely the Husimi function. Therefore Q is exactly th emeasurement outcome probability density for this coherent-state POVM.

Operationally, this means Q_ρ answers a different question than the Wigner function. While $W_\rho(q,p)$ encodes the full quantum state (including interference fringes and possible negativity), $Q_\rho(\alpha)$ encodes what a finite-resolution phase-space measurement can actually resolve.

5 Conclusion

This report has established a rigorous framework for the phase-space formulation of quantum mechanics, demonstrating that the abstraction of Hilbert space is not the singular path to describing quantum phenomena. By constructing the Weyl-Wigner correspondence, we have translated the operator algebra of observables and the density matrices of states into the geometric language of phase space. This translation reveals that quantum mechanics can be understood as a deformation of classical statistical mechanics, where the point-particle probability density is replaced by the Wigner quasi-probability distribution, and the commutative pointwise product is replaced by the noncommutative Moyal $\star$-product.

We have shown that the algebraic structure of quantum mechanics is encoded entirely within the $\star$-product. The derivation of the Moyal bracket confirmed that quantum dynamics are generated by a deformation of the Poisson bracket, providing a transparent view of the semiclassical limit where the classical equations of motion emerge as the leading-order approximation. Furthermore, the analysis of the harmonic oscillator and the canonical commutator demonstrated that this framework reproduces standard quantum results without recourse to wavefunctions.

Finally, we addressed the conceptual difficulty of the quantum-classical transition. Rather than treating the limit $\hbar \to 0$ as a mere mathematical formality, we interpreted it operationally through the lens of coarse-graining. We argued that macroscopic measurements act as low-pass filters on phase space, washing out the fine oscillatory structures that sustain the negativity of the Wigner function and the non-local corrections of the $\star$-product. Consequently, the emergence of classical reality is identified as an information-loss process, where the unique signatures of quantum algebra are suppressed by finite measurement resolution. Ultimately, the phase-space formulation provides an essential pedagogical bridge, unifying the distinct worlds of quantum and classical mechanics onto a single geometrical stage.

A Normalization Constant in Fourier Transform

Throughout this report we adopt the "physics" Fourier convention in which

$$\delta(z) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} d^d p \; e^{\frac{i}{\hbar} p \cdot z}. \tag{A.1}$$

With this convention, the **Weyl symbol** (inverse map) is taken to be

$$A_W(q,p) = \int_{\mathbb{R}^d} d^d z \; e^{-\frac{i}{\hbar} p \cdot z} \left\langle q + \frac{z}{2} \right| \hat{A} \left| q - \frac{z}{2} \right\rangle, \tag{A.2}$$

and we define the **Wigner function** as the Weyl symbol of the density operator, including the normalization factor

$$\boxed{W_\rho(q,p) := \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} d^d z \; e^{-\frac{i}{\hbar} p \cdot z} \left\langle q + \frac{z}{2} \right| \rho \left| q - \frac{z}{2} \right\rangle.} \tag{A.3}$$

This choice matches the convention used in the derivations that follow (in particular, the delta identity (A.1) is exactly what produces the prefactor in the phase-space pairing formula).

B Derivation of Weyl Quantization in Position Representation

In this appendix, we derive the position representation of the Weyl quantization formula. We start with the definition of the Weyl operator $\hat{A}$ given in Equation 2.1:

$$\hat{A} = \frac{1}{(2\pi\hbar)^{2d}} \int d^d q \, d^d p \, d^d \xi \, d^d \eta \, A(q, p) \exp\left\{\frac{i}{\hbar} [\xi(\hat{q} - q) + \eta(\hat{p} - p)]\right\}. \tag{B.1}$$

Our goal is to compute the matrix element $\langle x|\hat{A}|y\rangle$. Sandwiched between position eigenstates $\langle x|$ and $|y\rangle$, the expression becomes:

$$\langle x|\hat{A}|y\rangle = \frac{1}{(2\pi\hbar)^{2d}} \int dq \, dp \, d\xi \, d\eta \, A(q, p) e^{-\frac{i}{\hbar}(\xi q + \eta p)} \left\langle x \middle| \exp\left\{\frac{i}{\hbar}(\xi\hat{q} + \eta\hat{p})\right\} \middle| y \right\rangle. \tag{B.2}$$

To evaluate the matrix element of the exponential operator, we use the Baker-Campbell-Hausdorff (BCH) formula for two operators $\hat{X}$ and $\hat{Y}$ that commute with their commutator $[\hat{X}, \hat{Y}]$:

$$e^{\hat{X}+\hat{Y}} = e^{\hat{X}} e^{\hat{Y}} e^{-\frac{1}{2}[\hat{X},\hat{Y}]}. \tag{B.3}$$

Identifying $\hat{X} = \frac{i}{\hbar}\xi\hat{q}$ and $\hat{Y} = \frac{i}{\hbar}\eta\hat{p}$, we compute the commutator using the canonical relation $[\hat{q}, \hat{p}] = i\hbar$:

$$[\hat{X}, \hat{Y}] = \left(\frac{i}{\hbar}\right)^2 \xi\eta[\hat{q}, \hat{p}] = -\frac{1}{\hbar^2}\xi\eta(i\hbar) = -\frac{i}{\hbar}\xi\eta. \tag{B.4}$$

Substituting this into the BCH formula:

$$\exp\left\{\frac{i}{\hbar}(\xi\hat{q} + \eta\hat{p})\right\} = \exp\left(\frac{i}{\hbar}\xi\hat{q}\right) \exp\left(\frac{i}{\hbar}\eta\hat{p}\right) \exp\left(\frac{i}{2\hbar}\xi\eta\right). \tag{B.5}$$

Now we apply this to the state $|y\rangle$. Recall that $e^{\frac{i}{\hbar}\xi\hat{q}}$ is a phase operator in the position basis and $e^{\frac{i}{\hbar}\eta\hat{p}}$ acts as a translation operator, $e^{\frac{i}{\hbar}\eta\hat{p}}|y\rangle = |y - \eta\rangle$.

$$\begin{aligned} \left\langle x \middle| e^{\frac{i}{\hbar}\xi\hat{q}} e^{\frac{i}{\hbar}\eta\hat{p}} \middle| y \right\rangle &= \langle x|e^{\frac{i}{\hbar}\xi\hat{q}}|y - \eta\rangle \\ &= e^{\frac{i}{\hbar}\xi x}\langle x|y - \eta\rangle \\ &= e^{\frac{i}{\hbar}\xi x}\delta(x - y + \eta). \end{aligned} \tag{B.6}$$

Combining the phase factor from the BCH expansion, the full matrix element is:

$$\left\langle x \middle| \exp\left\{\frac{i}{\hbar}(\xi\hat{q} + \eta\hat{p})\right\} \middle| y \right\rangle = e^{\frac{i}{2\hbar}\xi\eta} e^{\frac{i}{\hbar}\xi x}\delta(x - y + \eta). \tag{B.7}$$

Substitute this result back into the integral for $\langle x|\hat{A}|y\rangle$:

$$\langle x|\hat{A}|y\rangle = \frac{1}{(2\pi\hbar)^{2d}} \int dq \, dp \, d\xi \, d\eta \, A(q, p) e^{-\frac{i}{\hbar}(\xi q + \eta p)} e^{\frac{i}{2\hbar}\xi\eta} e^{\frac{i}{\hbar}\xi x}\delta(x - y + \eta). \tag{B.8}$$

We can now integrate over η using the delta function $\delta(x - y + \eta)$, which implies $\eta = y - x$:

$$\langle x|\hat{A}|y\rangle = \frac{1}{(2\pi\hbar)^{2d}} \int dq \, dp \, d\xi \, A(q, p) \exp\left\{\frac{i}{\hbar}\left[-\xi q - (y - x)p + \frac{1}{2}\xi(y - x) + \xi x\right]\right\}. \tag{B.9}$$

Let us simplify the terms proportional to ξ in the exponent:

$$-\xi q + \frac{1}{2}\xi y - \frac{1}{2}\xi x + \xi x = \xi\left(\frac{x + y}{2} - q\right). \tag{B.10}$$

The integral becomes:

$$\langle x|\hat{A}|y\rangle = \frac{1}{(2\pi\hbar)^{2d}} \int dq\, dp\, A(q,p) e^{\frac{i}{\hbar}(x-y)p} \underbrace{\int d\xi\, \exp\left[\frac{i}{\hbar}\xi\left(\frac{x+y}{2} - q\right)\right]}_{(2\pi\hbar)^d \delta\left(q - \frac{x+y}{2}\right)}. \tag{B.11}$$

The ξ-integral yields a delta function enforcing $q = \frac{x+y}{2}$ (the midpoint rule). Integrating over q:

$$\langle x|\hat{A}|y\rangle = \frac{(2\pi\hbar)^d}{(2\pi\hbar)^{2d}} \int dp\, A\left(\frac{x+y}{2}, p\right) e^{\frac{i}{\hbar}(x-y)p}. \tag{B.12}$$

Simplifying the prefactor, we arrive at the final result for the position representation of the operator:

$$\langle x|\hat{A}|y\rangle = \frac{1}{(2\pi\hbar)^d} \int dp\, A\left(\frac{x+y}{2}, p\right) e^{\frac{i}{\hbar}(x-y)p}. \tag{B.13}$$

This confirms the correspondence between the Weyl symbol $A(q,p)$ and the operator kernel in position space.

C Derivation of the Differential Moyal $\star$-Product from the Integral Form

starting from the Fourier-space formula for the Weyl symbol of a product, we convert the extra "twist" phase into a bidifferential operator acting on the symbols.

Starting point

Assume we have reached the compact representation (Eq. (3.13) in the main text). The *first* exponential reconstructs the symbols from their Fourier transforms; the *second* exponential is the additional "twist" responsible for the deformation away from pointwise multiplication.

We also recall the inverse Fourier representations of the symbols:

$$A(p,q) = \int d^dX\, d^d\alpha\, \widetilde{A}(X,\alpha)\, \exp\left(\frac{i}{\hbar}(p\cdot X + q\cdot\alpha)\right), \tag{C.1}$$

$$B(p,q) = \int d^dY\, d^d\beta\, \widetilde{B}(Y,\beta)\, \exp\left(\frac{i}{\hbar}(p\cdot Y + q\cdot\beta)\right). \tag{C.2}$$

Step 1: Insert the series into the integral

Substitute (3.13) back into the integral (3.12). By linearity of integration, we may interchange the sum and the integral:

$$\begin{aligned} C(p,q) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i}{2\hbar}\right)^n \int d^dX\, d^dY\, d^d\alpha\, d^d\beta\, \widetilde{A}(X,\alpha)\, \widetilde{B}(Y,\beta)\, (\beta\cdot X - \alpha\cdot Y)^n \\ \times \exp\left(\frac{i}{\hbar}\big[q\cdot(\alpha+\beta) + p\cdot(X+Y)\big]\right). \end{aligned} \tag{C.3}$$

So the problem reduces to handling the polynomial factor $(\beta\cdot X - \alpha\cdot Y)^n$ inside a Fourier integral.

Step 3: Apply Fourier duality (the derivative trick)

The key Fourier identity is: *multiplying by a Fourier variable inside an inverse Fourier transform is equivalent to differentiating the reconstructed function.* From (C.1),

$$\frac{\partial}{\partial p_j} A(p,q) = \int d^dX\, d^d\alpha\, \widetilde{A}(X,\alpha)\left(\frac{i}{\hbar}X_j\right)\exp\left(\frac{i}{\hbar}(p\cdot X + q\cdot\alpha)\right),$$

hence (formally, under the integral sign)

$$X_j \longrightarrow \frac{\hbar}{i}\frac{\partial}{\partial p_j} \qquad \text{when acting on } A(p,q). \tag{C.4}$$

Similarly,

$$\alpha_j \longrightarrow \frac{\hbar}{i}\frac{\partial}{\partial q_j} \qquad \text{when acting on } A(p,q). \tag{C.5}$$

For $B(p,q)$, the same reasoning using (C.2) gives

$$Y_j \longrightarrow \frac{\hbar}{i}\frac{\partial}{\partial p_j}, \qquad \beta_j \longrightarrow \frac{\hbar}{i}\frac{\partial}{\partial q_j} \qquad \text{when acting on } B(p,q). \tag{C.6}$$

Left vs. right derivatives. In (C.3), the variables (X, α) belong to the Fourier integral that reconstructs A, whereas (Y, β) belong to the one that reconstructs B. To keep this bookkeeping explicit, we write

$\overleftarrow{\nabla}_p, \overleftarrow{\nabla}_q$ for derivatives acting on A (to the left), $\quad \overrightarrow{\nabla}_p, \overrightarrow{\nabla}_q$ for derivatives acting on B (to the right).

Then the four replacement rules become

$$X \to \frac{\hbar}{i}\overleftarrow{\nabla}_p, \quad \alpha \to \frac{\hbar}{i}\overleftarrow{\nabla}_q, \quad Y \to \frac{\hbar}{i}\overrightarrow{\nabla}_p, \quad \beta \to \frac{\hbar}{i}\overrightarrow{\nabla}_q. \tag{C.7}$$

Step 4: Substitute the operators into the polynomial

Apply (C.7) to the polynomial $(\beta \cdot X - \alpha \cdot Y)$:

$$\beta \cdot X \longrightarrow \left(\frac{\hbar}{i}\overrightarrow{\nabla}_q\right) \cdot \left(\frac{\hbar}{i}\overleftarrow{\nabla}_p\right) = \left(\frac{\hbar}{i}\right)^2 \overleftarrow{\nabla}_p \cdot \overrightarrow{\nabla}_q, \tag{C.8}$$

$$\alpha \cdot Y \longrightarrow \left(\frac{\hbar}{i}\overleftarrow{\nabla}_q\right) \cdot \left(\frac{\hbar}{i}\overrightarrow{\nabla}_p\right) = \left(\frac{\hbar}{i}\right)^2 \overleftarrow{\nabla}_q \cdot \overrightarrow{\nabla}_p. \tag{C.9}$$

Therefore

$$\boxed{(\beta \cdot X - \alpha \cdot Y) \longrightarrow \left(\frac{\hbar}{i}\right)^2 \left(\overleftarrow{\nabla}_p \cdot \overrightarrow{\nabla}_q - \overleftarrow{\nabla}_q \cdot \overrightarrow{\nabla}_p\right) = -\hbar^2\left(\overleftarrow{\nabla}_p \cdot \overrightarrow{\nabla}_q - \overleftarrow{\nabla}_q \cdot \overrightarrow{\nabla}_p\right).} \tag{C.10}$$

Step 5: Final assembly (series form, then exponential form)

Insert (C.10) into (C.3). Since the remaining Fourier integrals reconstruct $A(p, q)$ and $B(p, q)$, we obtain

$$\begin{aligned} C(p, q) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2\hbar}\right)^n A(p, q) \left[-\hbar^2\left(\overleftarrow{\nabla}_p \cdot \overrightarrow{\nabla}_q - \overleftarrow{\nabla}_q \cdot \overrightarrow{\nabla}_p\right)\right]^n B(p, q) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} A(p, q) \left[\frac{\hbar}{2i}\left(\overleftarrow{\nabla}_p \cdot \overrightarrow{\nabla}_q - \overleftarrow{\nabla}_q \cdot \overrightarrow{\nabla}_p\right)\right]^n B(p, q). \end{aligned} \tag{C.11}$$

(We used $\frac{i}{2\hbar}(-\hbar^2) = -\frac{i\hbar}{2} = \frac{\hbar}{2i}$.)

Recognising the Taylor series for the exponential, the sum becomes

$$\boxed{C(p, q) = A(p, q)\ \exp\left[\frac{\hbar}{2i}\left(\overleftarrow{\nabla}_p \cdot \overrightarrow{\nabla}_q - \overleftarrow{\nabla}_q \cdot \overrightarrow{\nabla}_p\right)\right]\ B(p, q).} \tag{C.12}$$

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